Ultra Scientist Vol. 25(1)A, 81-84 (2013).

Some Identities Involving Common Factors of Negafibonacci and Lucas Numbers

SANJAY HARNE[1], V.H. BADSHAH[2] and SAPNA SETHIYA[3]

(Acceptance Date 8th March, 2013)

Abstract

In this paper we present some identities involving common factors of Negafibonacci and Lucas numbers. Binet's formula of Negafibonacci will employ to obtain the identities.

Key words: Fibonacci numbers,Negafibonacci numbers, Binet's formula.

Mathematics subject classification 11B39, 11B37

1.Introduction

The Fibonacci and Lucas numbers appear in numerous mathematical problems. In Mathematical terms, the sequence of the Fibonacci numbers is defined by the recurrence relation[1,2].

$$F_n = F_{n-1} + F_{n-2}$$

with $F_o = 0,\ F_1 = 1$ where n≥1 (1.1)

F_n is called n^{th} Fibonacci number

Binet's formula is

$$F_n = \frac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{2^n\sqrt{5}} = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad (1.2)$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ (1.3)

which gives $\alpha + \beta = 1$ and $\alpha.\beta = -1$

The sequence can also be extended to negative index n using the re-arranged recurrence relation -

$$F_{n-2} = F_n - F_{n-1} \quad (1.4)$$

Binet's formula of "Negafibonacci" number is given by

$$F_{-n} = (-1)^{n+1}\left[\frac{\alpha^n - \beta^n}{\alpha - \beta}\right] \quad (1.5)$$

The Lucas sequence is defined[3] as

$L_{n+1} = L_n + L_{n-1}$, where n≥1 with L_0= 2, L_1=1 (1.5)

Binet's formula for the Lucas sequence is

$$L_n = \frac{\left(1+\sqrt{5}\right)^n + \left(1-\sqrt{5}\right)^n}{2^n} = \alpha^n + \beta^n \quad (1.6)$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ (3)

which gives $\alpha + \beta = 1$ and $\alpha.\beta = -1$

2. Identities for Common Factors of Negafibonacci and Lucas Number :

Theorem 2.1 $F_{8n}+1=F_{-(4n-1)}\ L_{4n+1}$, where $n \geq 1$

Proof : $F_{-(4n-1)} L_{4n} = (-1)^{4n}\left[\frac{\alpha^{4n-1} - \beta^{4n-1}}{\alpha - \beta}\right]$

$$\left[\alpha^{4n+1} + \beta^{4n+1}\right] \quad \{\text{By } (1.5)\}$$

$$= \left[\frac{\alpha^{8n} - \beta^{8n}}{\alpha - \beta}\right] + \left[\frac{\alpha^{4n-1}.\beta^{4n+1} - \beta^{4n-1}.\alpha^{4n+1}}{\alpha - \beta}\right]$$

$$= \left[\frac{\alpha^{8n} - \beta^{8n}}{\alpha - \beta}\right] + \left[\frac{\alpha^{4n-1}.\beta^{4n+1} - \beta^{4n-1}.\alpha^{4n+1}}{\alpha - \beta}\right]$$

$$= F_{8n} + (-1)^{4n}(\alpha + \beta) \quad \{\text{By}(1.2)\text{and}(1.3)\}$$

$$= F_{8n} + 1$$

Theorem 2.2 $F_{8n+1} + 1 = F_{-(4n+1)}.L_{4n}\ \ n \geq 1$

$$\textit{Proof :}\ F_{-(4n-1)}.L_{4n} = (-1)^{4n+2}\left[\frac{\alpha^{4n-1} - \beta^{4n-1}}{\alpha - \beta}\right].\left[\alpha^{4n} + \beta^{4n}\right] \quad (\text{By } (1.5)$$

$$= \left[\frac{\alpha^{8n+1} - \beta^{8n+1}}{\alpha - \beta}\right] + \frac{1}{\alpha - \beta}\left[\alpha^{4n-1}.\beta^{4n} - \beta^{4n+1}.\alpha^{4n}\right]$$

$$= F_{8n+1} + \frac{1}{\alpha - \beta}(-1)^{4n}\left[\alpha - \beta\right] \quad \{\text{By (1.2) and (1.3)}\}$$

$$= F_{8n+1} + 1$$

Theorem 2.3 $F_{8n+2} + 1 = F_{-(4n+2)}.L_{4n}$, where $n \geq 1$

$$\textit{Proof:}\ F_{-(4n+2)}.L_{4n} = (-1)^{4n+3}\left[-\frac{\alpha^{4n+2} - \beta^{4n+2}}{\alpha - \beta}\right].\left[\alpha^{4n} + \beta^{4n}\right] \quad \text{By (1.5)}$$

$$= (-1)^{4n+3}\left[\frac{\alpha^{8n+2}-\beta^{8n+2}}{\alpha-\beta}\right]+\frac{1}{\alpha-\beta}\left[\alpha^{4n+2}.\beta^{4n}-\alpha^{4n}.\beta^{4n+2}\right]$$

$$= -\left[F_{8n+2}+\frac{1}{\alpha-\beta}(-1)^{4n}\left[\alpha^2-\beta^2\right]\right] \qquad \{\text{By (1.2) and (1.3)}\}$$

$$= F_{8n+3}+1$$

By the same way we have the following Result

Lemma 2.5 $F_{-2n}.L_{2n} = -F_{4n}$

Proof : $F_{-2n}\,L_{2n} = (-1)^{2n+1}\left[\frac{\alpha^{2n}-\beta^{2n}}{\alpha-\beta}\right]$ By (1.5)

$$= -\left[\frac{\left(\alpha^{4n}-\beta^{4n}\right)}{\alpha-\beta}\right]$$

$$= -F_{4n}$$

Lemma 2.6 $-(F_{8n+1}-1) = F_{-4n}.L_{4n+1}$ where $n \geq 1$

Proof: $F_{-4n}.L_{4n+1} = (-1)^{4n+1}\left[\frac{\alpha^{4n}-\beta^{4n}}{\alpha-\beta}\right]\left[\alpha^{4n+1}+\beta^{4n+1}\right]$ By (1.5)

$$= \left[\left(\frac{\alpha^{8n}-\beta^{8n}}{\alpha-\beta}\right)+\frac{1}{\alpha+\beta}\alpha^{4n}.\beta^{4n+1}-\beta^{4n+1}.\alpha^{4n}\right]$$

$$= F_{8n+1}+\frac{(-1)^{4n}}{\alpha-\beta}(\beta-\alpha) \qquad \{\text{By (1.2) and (1.3)}\}$$

$$= -\left[F_{8n+1}-1\right] = -F_2.L_{2n}L_{4n+1}$$

From **Lemma 5** and **Lemma 6**, we have the following result,

Corollary 2.7 $-(F_{8n+1}-1) = -F_{2n}.L_{2n}L_{4n+1}$

Lemma 2.8 $-\left(F_{8n+3}-1\right)=-F_{-(4n+2)}.L_{4n+1}$, where $n \geq 1$

Theorem 2.9 $L_{8n+1}+1=-L_{4n}.L_{4n+1}$ where $n \geq 1$

Theorem 2.10 $L_{8n+3}-1=L_{4n+1}.L_{4n+2}$ where $n \geq 1$

Theorem 2.11 $L_{4n+3}+1=5\,L_{2n+1}.F_{2n+1}.F_{4n+1}$, $n \geq 0$

References

1. H. Dubner and W. Keller, New Fibonacci and Lucas Primes, *Mathematics of Computation, 68,* 417-427 (1999).
2. M. Thongmoon, Identities for the Common Factors of Fibonacci and Lucas numbers, *Int. Math. Forum, 4,* no.*7,* 303-308 (2009).
3. T. Koshy, Fibonacci and Lucas Numbers with Applications, A.Wiley-Interscience Publication, New York (2001).