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LRS Bianchi Type II Bulk Viscous Stiff Fluid Cosmological Models with Variable G and Λ

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Abstract

We have investigated the Bianchi type II bulk viscous stiff fluid cosmological models with variable gravitational constant *G* and cosmological constant Λ. To obtain the deterministic model, we have assumed that the expansion θ in the model is proportional to the shear σ. Various physical and geometrical features of the models are also discussed.

Key words : Bianchi Type-II, Stiff fluid, Bulk viscosity, Variable Λ, Variable G.

1. Introduction

The Einstein field equation has two parameters, the gravitational constant G and the cosmological constant Λ. The Newtonian constant of gravitation G plays the role of a coupling constant between geometry and matter in the Einstein field equation. In the evolving universe, it is natural to look at this constant as a function of time. Dirac[14] first introduce the idea of variable G in what he called his Large Number Hypothesis (LNH) and since various work has been done for a modified general relativity theory with this variation in G. An important modification has come into picture, which links the variation of G with that of the variable Λ term within the framework of general theory of relativity. An appealing feature of this modification is that it leaves the form of Einstein field equation formally unchanged by allowing the variation of G accompanied by a change in Λ and enables us to solve many cosmological problem such as the cosmological constant problem, inflationary scenario etc.[26]. This approach is first proposed by Lau[19] and since many authors[21,27] have investigated cosmological models with variable G and Λ. Bianchi type I cosmological model with variable G and Λ have been studied by Singh and Agrawal[28], Beesham[7], Kalligas *et.al*.[15], Arbab[2,3], Beesham *et.al*.[8] Kilinc[18], Vishwakarma[33], Chakraborty and Roy[12] Singh *et.al*.[30].

In recent years, an appealing idea[1,9,10,16,22,30] has been developed to consider joint variation of G and Λ within the framework of general relativity. Arbab[4] has considered cosmological models with viscous fluid considering G and Λ. A lot of work has been done by Saha[23-25] in studying the anisotropic Bianchi type I cosmological model in general relativity with varying G and Λ. In recent years, there has been considerable interest in Bianchi type II cosmology. Bianchi type II models play an important role in current modern cosmology, for simplification and description of the large scale behavior of the actual universe. The present day universe is satisfactorily be described by homogeneous and isotropic models given by FRW space-time. The universe in smaller scale is neither homogeneous nor isotropic and we do not expect the universe in its early stages to have these properties. Homogeneous and isotropic cosmological models have been studied in the framework of general relativity in the search of realistic picture of the universe in its early stages. Although these are more restricted than the homogeneous models which explain a number of observed phenomena quite satisfactorily. Bianchi type II space time has a fundamental role in constructing cosmological models suitable for describing the early stages of evolution of universe. Asseo and Sol[5] emphasized the importance of Bianchi type II universe. Chakraborty and Roy[11] have investigated Bianchi type II cosmological model with variable G and Λ.

Motivated by the situations discussed above, in this paper we have investigated homogeneous Bianchi type II space time with variable G and ▯ containing matter in form of bulk viscous stiff-fluid. To get a determinate solution, we have used a supplementary condition $R = m S^n$ between the metric potentials. The physical and geometrical implications of the models are also discussed.

2. The metric and field equations :

We consider the LRS Bianchi type-II metric in the form

$$ds^2 = -dt^2 + R^2\left(dx^2 + dz^2\right) + S^2\left(dy - x dz\right)^2, \quad (1)$$

where the metric potentials $R \, and \, S$ are functions of time alone and $\sqrt{-g} = R^2 S$.

We assume that the cosmic matter consists of bulk viscous fluid given by

$$T^i_j = \left(\rho + \bar{p}\right) v_j v^i + \bar{p} g^i_j, \quad (2)$$

where, ρ is energy density, $\bar{p}$ is effective pressure and v^j the four velocity vector satisfying the relation satisfying $v_i v^j = -1. \quad (3)$

The effective pressure $\bar{p}$ is related to the equilibrium pressure p is given by [6]:

$$\bar{p} = p - 3\eta H, \quad (4)$$

where $\eta = \eta_0 \rho^b$ [6] stand for coefficient of bulk viscosity.

The Einstein's field equations $(C = 1)$ with time dependent gravitational and cosmological term are

$$R_i^j - \frac{1}{2} R g_i^j = -8\pi \mathrm{G} T_i^j + \Lambda g_i^j, \tag{5}$$

we assume the co-ordinates to be co-moving so that

$$v^i = (0,0,0,1) \quad x^i = (\frac{1}{R}, 0,0,0) \tag{6}$$

In the co-moving system of co-ordinates the field equation (5) for the metric (1) and the matter distribution (2) lead to the following system of equations

$$\frac{R_{44}}{R} + \frac{S_{44}}{S} + \frac{R_4 S_4}{RS} + \frac{S^2}{4R^4} = -8\pi \mathrm{G}\,\bar{\mathrm{p}} + \Lambda, \tag{7}$$

$$2\frac{R_{44}}{R} + \frac{R_4^2}{R} - \frac{3S^2}{4R^4} = -8\pi \mathrm{G}\,\bar{\mathrm{p}} + \Lambda, \tag{8}$$

$$\frac{R_4^2}{R^2} + 2\frac{R_4 S_4}{RS} - \frac{S^2}{4R^4} = 8\pi \mathrm{G}\,\rho + \Lambda, \tag{9}$$

the suffix 4 by the symbols *R* and *S* denotes a differentiation with respect to *t*.

An additional equation for time changes of G and Λ is obtained by the divergence of Einstein tensor, *i.e.* $\left(R_i^j - \frac{1}{2} R g_i^j\right)_{;j} = 0$, which leads to $\left(8\pi \mathrm{G} T_i^j - \Lambda g_i^j\right)_{;j} = 0$, then

$$8\pi G_4 \rho + \Lambda_4 + 8\pi G\left[\rho_4 + \left(\rho + \bar{p}\right)\left(2\frac{R_4}{R} + \frac{S_4}{S}\right)\right] = 0, \tag{10}$$

The energy conservation equation $\left(T_i^j\right)_{;j} = 0$, after using eq. (4), splits (10) into two equations

$$\rho_4 + (\rho + p)\left(2\frac{R_4}{R} + \frac{S_4}{S}\right) = 0, \tag{11}$$

$$8\pi G_4 \rho + \Lambda_4 = 8\pi G(3\eta H)\left(2\frac{R_4}{R} + \frac{S_4}{S}\right), \tag{12}$$

Equation (10) together with the energy conservation equation leads to

$$8\pi G_4 \rho + \Lambda_4 = 0, \tag{13}$$

Implying that Λ is constant whenever G is constant.

The scalar of expansion θ is given by

$$\theta = \left(2\frac{R_4}{R} + \frac{S_4}{S}\right), \tag{14}$$

The shear (σ) is given by

$$\sigma^2 = \frac{1}{2}\left[\left(\sigma_1^1\right)^2 + \left(\sigma_2^2\right)^2 + \left(\sigma_3^3\right)^2 + \left(\sigma_4^4\right)^2\right], \tag{15}$$

therefore

$$\sigma^2 = \frac{1}{3}\left[\frac{R_4}{R} - \frac{S_4}{S}\right]^2, \tag{16}$$

The proper volume (V^3) is given by

$$V^3 = R^2 S, \tag{17}$$

The Hubble parameter (H) is given by

$$H = \frac{1}{3}\left[2\frac{R_4}{R} + \frac{S_4}{S}\right], \tag{18}$$

3. Solutions of the field equations :

The field equations (7)-(9) are a system of three equations in six unknowns *R*, *S*, Λ, G, $\bar{p}$, ρ. To obtain a deterministic solution, we assume that the expansion θ is proportional to the shear σ as in [34], which leads to

$$R = mS^n, \tag{19}$$

The motive behind assuming this condition is explained with reference to Thorne[31,32], the observation of the velocity-red-shift relation for extragalactic source suggest by the Hubble expansion of the universe is isotropic today within ≈ 30 per cent[17]. To put more precisely, red-shift studies place the limit

$$\frac{\sigma}{H} \le 0.3$$

on the ratio of shear to Hubble constant in the neighborhood of our galaxy today. Collins *et.al*.[13] have pointed out that for spatially homogeneous metric, the normal congruence to the homogeneous expansion satisfies the condition $\frac{\sigma}{H}$ = constant.

From Equations (7) and (8) we have

$$\frac{R_{44}}{R} + \frac{R_4^2}{R^2} - \frac{S_{44}}{S} - \frac{R_4 S_4}{RS} - \frac{S^2}{R^4} = 0, \tag{20}$$

Equations (19) and (20) together lead to

$$\frac{S_{44}}{S} + 2n\frac{S_4^2}{S^4} = \frac{S^{2-4n}}{(n-1)m^4}, \tag{21}$$

Equation (21) leads to

$$2S_{44} + \frac{4n}{S}S_4^2 = \frac{2S^{3-4n}}{(n-1)m^4}, \tag{22}$$

Equation (22) leads to

$$\frac{df^2}{dS} + \frac{4n}{S}f^2 = \frac{2S^{3-4n}}{(n-1)m^4}, \tag{23}$$

where $S_4 = f(S), S_{44} = ff' \, and \, f' = \frac{df}{ds}$,

Equation (23) leads to

$$f^2 = \frac{1}{2(n-1)m^4}S^{4(1-n)} + \alpha S^{-4n}, \tag{24}$$

where α is constant of integration. Equation (24) leads to

$$\frac{S^{2n}\,ds}{\sqrt{\alpha + \beta S^4}} = dt$$
$$where\ \beta = \frac{1}{2(n-1)m^4} \tag{25}$$

the solution of (25) is not tenable for n = 1. For simplification and realistic visualization of the exact behavior of our universe, we investigate the model taking such values of n so that the non linear differential equation (25) can exactly be solved. We consider the following two cases[29].

4. First model: $n = \frac{1}{2}$

In this case equation (25) reduces to

$$\frac{m^2 S\,ds}{\sqrt{\alpha m^4 - S^4}} = dt, \tag{26}$$

which on integration leads to

$$S^2 = m^2\sqrt{\alpha}\,Sin\left(\frac{2t}{m^2} + \gamma\right), \tag{27}$$

where γ is constant of integration. Hence equations (19) and (27) lead to

$$R^2 = m^3\alpha^{\frac{1}{4}}\left[Sin\left(\frac{2t}{m^2} + \gamma\right)\right]^{\frac{1}{2}}, \tag{28}$$

Substitute the values from (27) and (28) in metric (1), we get

$$ds^2 = -dt^2 + m^3 \alpha^{\frac{1}{4}} \sqrt{Sin\left(\frac{2t}{m^2}+\gamma\right)}\left[dx^2+dz^2\right] + m^2\sqrt{\alpha}\, Sin\left(\frac{2t}{m^2}+\gamma\right)\left[dy - xdz\right]^2, \quad (29)$$

After applying suitable transformation of co-ordinates the metric (29) reduces to the form

$$ds^2 = -\frac{m^4}{4} dT^2 + m^3 \alpha^{\frac{1}{4}} \sqrt{SinT}\left[dX^2+dZ^2\right] + m^2 \sqrt{\alpha}\, SinT\left[dY - XdZ\right]^2.$$
$$where\ \frac{2t}{m^2}+\gamma = T, x = X,\ y = Y, z = Z \quad (30)$$

5. Some physical and geometrical features:

The important physical quantities like scalar of expansion (θ), shear scalar (σ), proper volume (v), and the Hubble parameter (H) for the model (30) are given by

$$\theta = \frac{2\cot T}{m^2}, \quad (31)$$

$$\sigma = \frac{\cot T}{2\sqrt{3}\, m^2}, \quad (32)$$

$$H = \frac{2\cot T}{3m^2}, \quad (33)$$

$$V^3 = m^4 \sqrt{\alpha}\, SinT. \quad (34)$$

The ratio of shear and scalar of expansion is calculated as

$$\frac{\sigma}{\theta} = \frac{1}{4\sqrt{3}}, \quad (35)$$

which proves our assumption.
Also assume that the matter content is stiff fluid which leads to

$$p = \rho \quad (36)$$

Equation (11) together with (19) and (36) after integration leads to

$\rho = \frac{k}{S^4}$, where k being constant (37)

Differentiating (37), we obtain

$$\rho_4 = \frac{-4k}{S^4}\left(\frac{S_4}{S}\right) \quad (38)$$

Now with help of (19) and (27) equation (9) becomes

$$8\pi G\rho + \Lambda = \frac{5\cot^2 T}{4m^4} - \frac{1}{4m^4}, \quad (39)$$

On differentiation of Eq. (39), we obtain

$$8\pi G_4\, \rho + 8\pi G\rho_4 + \Lambda_4 = -\frac{5\cot T\, cosec^2 T}{m^6}, \quad (40)$$

with help of Eqs. (12), (19) and (27) Eq. (40) becomes

$$8\pi G\left[\rho_4 + 4\eta\frac{\cot^2 T}{m^4}\right] = -\frac{5\cot T\, cosec^2 T}{m^6}, \quad (41)$$

Again substituting (38) into (41),we get

$$8\pi G\left[\frac{4k\cot T}{m^6\alpha\, Sin^2 T}-\frac{4\eta\cot^2 T}{m^4}\right]=\frac{5\cot T\cos ec^2 T}{m^6}, \tag{42}$$

Equation (42) leads to

$$G=\frac{5\cos ec^2 T}{32\pi}\left[\frac{k}{\alpha\, Sin^2 T}-\eta\, m^2\cot T\right]^{-1}, \tag{43}$$

Now substituting eqs. (37) and (43) into eq. (39), we get

$$\Lambda=\frac{1}{4m^4}\left[5\cot^2 T-1-\frac{5k\cos ec^2 T}{\left(k-\eta\, m^2\alpha\,\sin^2 T\cot T\right)}\right], \tag{44}$$

where η is given by

$$\eta=\eta_0\left[\frac{k}{m^4\,\alpha\, Sin^2\, T}\right]^b, \tag{45}$$

The pressure and density of the model (30) are given by

$$p=\rho=\frac{k}{m^4\,\alpha\, Sin^2\, T}. \tag{46}$$

In this case, it is observed that the spatial volume is zero and the expansion scalar infinite at T= 0, *i.e.* at $t=-\frac{1}{2}\gamma\, m^2$. This shows that the evolution of the universe starts with zero volume and the rate of expansion is infinite at T = 0. As T increases, spatial volume V increases, but the rate of expansion slows down. However, the expansion in the model stops when $T=\frac{\pi}{2}$. Thus the model in general represents an expanding, shearing and non rotating universe. The energy density (ρ), pressure (p) are infinite at the initial singularity T= 0, *i.e.* at $t=-\frac{1}{2}\gamma\, m^2$. We observe that for $T=\frac{\pi}{2}$, $\Lambda=-\frac{3}{2m^4}$, which is a constant. Also in this case as $T\to 0, \Lambda\to\infty$. There is a point type singularity in the model at T= 0 [20].

6. Second model: $n=\frac{3}{2}$

In this case equation (25) reduces to

$$\frac{m^2 S^3\, ds}{\sqrt{\alpha\, m^4+S^4}}=dt, \tag{47}$$

which on integration leads to

$$S^2=\left[\left\{\frac{2(t+\delta)}{m^2}\right\}^2-\alpha\, m^4\right]^{\frac{1}{2}}, \tag{48}$$

where δ is constant of integration. Hence equations (19) and (48) lead to

$$R^2=m^2\left[\left\{\frac{2(t+\delta)}{m^2}\right\}^2-\alpha\, m^4\right]^{\frac{3}{4}}, \tag{49}$$

Substitute the values from (48) and (49) in metric (1), we get

$$ds^2 = -\,dt^2 + m^2\left[\left\{\frac{2(t+\delta)}{m^2}\right\}^2 - \alpha\, m^4\right]^{\frac{3}{4}}\left[dx^2 + dz^2\right] + \left[\left\{\frac{2(t+\delta)}{m^2}\right\}^2 - \alpha\, m^4\right]^{\frac{1}{2}}\left[dy - xdz\right]^2, \tag{50}$$

After applying suitable transformation of co-ordinates the metric (50) reduces to the form

$$ds^2 = -\frac{m^4}{4}dT^2 + m^2\left[T^2 - \alpha\, m^4\right]^{\frac{3}{4}}\left[dX^2 + dZ^2\right] + \left[T^2 - \alpha\, m^4\right]^{\frac{1}{2}}\left[dY - XdZ\right]^2.$$
$$where\ \frac{2(t+\delta)}{m^2} = T, x = X,\ y = Y, z = Z \tag{51}$$

7. Some physical and geometrical features:

The important physical quantities like scalar of expansion (θ), shear scalar (σ), proper volume (v), and the Hubble parameter (H) for the model (51) are given by

$$\theta = \frac{4T}{m^2\left[T^2 - \alpha\, m^4\right]}, \tag{52}$$

$$\sigma = \frac{T}{4\sqrt{3}\,m^2\left[T^2 - \alpha\, m^4\right]}, \tag{53}$$

$$\mathrm{H} = \frac{4T}{3m^2\left[T^2 - \alpha\, m^4\right]}, \tag{54}$$

$$\mathrm{V}^3 = m^2\left[T^2 - \alpha\, m^4\right]. \tag{55}$$

The ratio of shear and scalar of expansion is calculated as

$$\frac{\sigma}{\theta} = \frac{1}{8\sqrt{3}}, \tag{56}$$

which proves our assumption.

Equation (11) together with (19) and (36) after integration leads to

$$\rho = \frac{k}{S^8}\text{ , where } k \text{ being constant} \tag{57}$$

Differentiating (57), we obtain

$$\rho_4 = \frac{-8k}{S^8}\left(\frac{S_4}{S}\right) \tag{58}$$

Now with help of (19) and (48) equation (9) becomes

$$8\pi \mathrm{G}\,\rho + \Lambda = \frac{\left[20T^2 + \alpha\, m^4\right]}{4m^4\left[T^2 - \alpha\, m^4\right]^2}, \tag{59}$$

On differentiation of eq. (59), we obtain

$$8\pi G_4\,\rho + 8\pi G\,\rho_4 + \Lambda_4 = -\frac{2T\left[10T^2 + 11\alpha m^4\right]}{m^6\left[T^2 - \alpha m^4\right]^3}, \tag{60}$$

with help of Eqs. (12), (19) and (48) eq. (60) becomes

$$8\pi G\left[\rho_4 + 16\eta\frac{T^2}{m^4\left[T^2 - \alpha m^4\right]^2}\right] = -\frac{2T\left[10T^2 + 11\alpha m^4\right]}{m^6\left[T^2 - \alpha m^4\right]^3}, \tag{61}$$

Again substituting (58) into (61), we get

$$8\pi G\left[\frac{-8kT}{m^2\left[T^2 - \alpha m^4\right]^3} + 16\eta\frac{T^2}{m^4\left[T^2 - \alpha m^4\right]^2}\right] = -\frac{2T\left[10T^2 + 11\alpha m^4\right]}{m^6\left[T^2 - \alpha m^4\right]^3}, \tag{62}$$

Equation (62) leads to

$$G = \frac{\left[10T^2 + 11\alpha m^4\right]}{32\pi m^4\left[T^2 - \alpha m^4\right]}\left[\frac{k}{\left[T^2 - \alpha m^4\right]} - \frac{2\eta T}{m^2}\right]^{-1}, \tag{63}$$

Now substituting Eq. (57) and (63) into eq. (59), we get

$$\Lambda = \frac{1}{4m^4\left[T^2 - \alpha m^4\right]^2}\left[\left(20T^2 + \alpha m^4\right) - m^2 k\frac{\left(10T^2 + 11\alpha m^4\right)}{m^2 k - 2\eta T\left(T^2 - \alpha m^4\right)}\right], \tag{64}$$

where η is given by

$$\eta = \eta_0\left[\frac{k}{\left(T^2 - \alpha m^4\right)^2}\right]^b, \tag{65}$$

The pressure and density of the model (51)

$$p = \rho = \frac{k}{\left(T^2 - \alpha m^4\right)^2}. \tag{67}$$

In this case the universe starts with an infinite rate of expansion from $T = m^2\sqrt{\alpha}$. Initially at this time the spatial volume is zero. At $T = 0$, the other physical quantities like scalar of expansion (θ), shear scalar (σ) and the Hubble parameter (H) becomes zero and at $T = m^2\sqrt{\alpha}$, these physical quantities become infinite. The energy density (ρ), pressure (p) are infinite at the initial singularity $T = m^2\sqrt{\alpha}$. We observe that under suitable consideration of constant m and k, the cosmological term decreases as time increases and it approaches small positive value at late time. There is a point type singularity in the model at $T = m^2\sqrt{\alpha}$ [20]. The Gravitational constant *G(t)* is gradually decreases and tends to zero at late time.

8. Conclusion

The models obtain in this paper start with big bang singularity evolving zero volume with infinite rate of expansion. The presence of a point type singularity is observed in the both models. In general, both the models represent expanding, shearing and non rotating universe. The physical and kinematical parameters like scalar expansion (θ), shear (σ) and the Hubble parameter (H) decrease with time in both the models. The spatial volume (*V*) increase with time. Since $\frac{\sigma}{\theta}$ = constant, hence the anisotropy is maintained throughout in both the models and it gives essentially an empty space for large time.

The cosmological constant is found to be decreasing function of cosmic time and it approaches to a small value at late time. The gravitational constant decreases with time and tends to zero. It is also remarkable to mention that the model (30) involves periodic functions, which gives rise to a cyclic mode of expansion.

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