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# Analytical solution of Burgers-like equation

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## Abstract

In this paper we determine optimal system for one-dimensional Burgers-like equation,

$$c_t = [V(c)c]_x + [D(c)c_x]_x$$

Lie group analysis is used to obtain invariant vector fields of one-dimensional Burgers-like equation. These invariant vector fields forms the Lie algebra. Certain choice of invariant vector field defines the transformation to convert the equation into a solvable partial differential equation. With help of adjoint representation table one-dimensional optimal system is obtained.

*Key words:* invariant vector fields, optimal system of Lie subalgebras.

## I. Introduction

Due to effect of gravity upon the particles, if they are heavier than the surrounding fluid the resulting motion is called sedimentation[4]. The concentrat- ion profile in the dilute limit $c(x,t)$ obeys a Burgers-like equation[1],

$$c_t = [V(c)c]_x + [D(c)c_x]_x \qquad (1)$$

Where, $V(c)$ and $D(c)$ represent the Stokes velocity and gradient diffusivity modified by the presence of other particles. According to Batchelor calculations[2], in dilute limit $c \ll 1$, $V(c)$ is linear in c and gradient diffusivity $D(c)$ can be taken as constant. So the equation (1) becomes,

$$c_t = (2ac + b)c_x + \beta\, c_{xx} \qquad (2)$$

In this paper we present solutions of equation (2). Since equation is non-linear, it is not possible to list all the solutions and therefore the one-dimensional optimal system is determined. The construction of one-dimensional optimal system gives the set of vector fields. These vector fields generate non-equivalent solutions. The paper is organized as follows. In section II basic definitions and theorems on Lie group theory are presented. Section III is devoted to determine invariant vector fields. These invariant vector fields are used in section IV to determine certain analytical solutions of equation (2). In section V, adjoint representation table is constructed and is used to determine one-dimensional optimal system. Conclusions are given at the end.

*II. Preliminaries:*

In this section we state the basic definitions and theorems on symmetry group and Lie algebra[3].

*i) The General Prolongation formula:* Let

$$v = \sum_{i=1}^{p} \xi^i (x,\mathrm{u}) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha (x,\mathrm{u}) \frac{\partial}{\partial u^\alpha}$$

be a vector field defined on an open subset $M \subset X\ U$. The n[th] prolongation of is the vector field

$$pr^{(n)}v = v + \sum_{\alpha=1}^{q} \sum_{j} \phi_\alpha^j(x,u^{(n)}) \frac{\partial}{\partial u_j^\alpha} \quad (3)$$

defined on the corresponding jet space $M^{(n)} \subset X \times U^{(n)}$, the second summation being over all multi-indices $j = (j_1, j_2, j_3, \ldots\ldots, j_k)$, with $1 \le j_k \le p$, $1 \le k \le n$. The coefficient functions $\phi_\alpha^j$ of $pr^{(n)}v$ are given by the formula:

$$\phi_\alpha^j(x,u^{(n)}) = D_j\left(\phi_\alpha - \sum_{i=1}^{p} \xi^i u_i^\alpha\right) + \sum_{i=1}^{p} \xi^i u_{j,i}^\alpha$$

where, $u_i^\alpha = \dfrac{\partial u^\alpha}{\partial x^i}$ and $u_{j,i}^\alpha = \dfrac{\partial u_j^\alpha}{\partial x^i}$.

*ii) Theorem 1:* Suppose $\Delta_\vartheta(x,u^{(n)}) = 0$, $\vartheta = 1, \ldots\ldots\ldots, \mathrm{l}$, is a system of differential equations of maximal rank defined over $M \subset X \times U$. If G is a local group of transformations acting on M, and

$$pr^{(n)}v[\Delta_\vartheta(x,u^{(n)})] = 0,$$

$\vartheta = 1, \ldots\ldots, \mathrm{l}$, whenever $\Delta_\vartheta(x,u^{(n)}) = 0$,

for every infinitesimal generator $v$ of G, then G is a symmetry group of the system.

*III. Invariant vector fields:*

In this section we determine invariant vector fields for equation (2). On applying theorem 1 and general prolongation formula (3), we determine the symmetry group of equation (2). Suppose the symmetry group of equation (2) is generated by vector field,

$$v = \tau(t,x,c)\frac{\partial}{\partial t} + \xi(t,x,c)\frac{\partial}{\partial x} + \phi(t,x,c)\frac{\partial}{\partial c}$$

On applying the second prolongation of vector field on equation (2) we get,

$$\phi^t - (2ac+b)\phi^x - 2ac_x\phi - \beta\phi^{xx} = 0 \quad (5)$$

where,

$$\phi^t = \phi_t - \xi_t c_x + (\phi_c - \tau_t)c_t - \xi_c c_x c_t - \tau_c c_t^2$$

$$\phi^x = \phi_x + (\phi_c - \xi_x)c_x - \tau_x c_t - \xi_c c_x^2 - \tau_c c_x c_t$$

$$\phi^{xx} = \phi_{xx} + (2\phi_{xc} - \xi_{xx})c_x - \tau_{xx}c_t + (\phi_{cc} - 2\xi_{xc})c_x^2 - 2\tau_{xc}c_x c_t - \xi_{cc}c_x^3$$

$$- \tau_{cc}c_x^2c_t + (\phi_c - 2\xi_x)c_{xx}$$
$$- 2\tau_x c_{xt} - 3\xi_c c_x c_{xx} - \tau_c c_t c_{xx}$$
$$- 2\tau_c c_x c_{xt}$$

The generators $\tau, \xi, \phi$ are functions of $t$, $x$, $c$ and are independent of partial derivatives of c with respect to $t$ and $x$. On substituting the expression for $\phi^t, \phi^x, \phi^{xx}$ in equation (5) and on equating the coefficients of equal powers of partial derivatives of $c$ w.r.t. $x$ and $t$ we get the following system of equations.

$$\tau_x = \tau_c = 0, \xi_c = 0$$
$$2\xi_x - \tau_t = 0$$
$$\beta\phi_{cc} = 0$$
$$-\xi_t + (2ac + b)(\xi_x - \tau_t) - 2a\phi - \beta(2\phi_{xc} - \xi_{xx}) = 0$$
$$\phi_t - (2ac + b)\phi_x - \beta\phi_{xx} = 0$$

On simultaneous evaluation of above system of equations we get,

$$\tau = C_1 + tC_4 + t^2C_5.$$
$$\xi = C_2 + tC_3 + \frac{1}{2}xC_4 + txC_5.$$
$$\phi = -\frac{1}{2a}C_3 - \left(\frac{1}{2}c + \frac{b}{4a}\right)C_4 + \left(-tc - \frac{1}{2a}x - \frac{b}{2a}\,t\right)C_5.$$

where $C_1, C_2, \ldots, C_5$ are arbitrary constants. With respect to each constant $C_i$ we get an invariant vector field. Thus the Lie algebra is generated by the following five vector fields.

$$v_1 = \frac{\partial}{\partial t}$$
$$v_2 = \frac{\partial}{\partial x}$$
$$v_3 = t\frac{\partial}{\partial x} - \frac{1}{2a}\frac{\partial}{\partial c}$$
$$v_4 = t\frac{\partial}{\partial t} + \frac{1}{2}\,x\frac{\partial}{\partial x} + \left(-\frac{1}{2}c - \frac{b}{4a}\right)\frac{\partial}{\partial c}$$
$$v_5 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + \left(-tc - \frac{1}{2a}\,x - \frac{b}{2a}\,t\right)\frac{\partial}{\partial c}$$

The commutation relations between vector fields are given by the following table 1. The entry in row $i$ and column $j$ representing $[v_i, v_j]$ and is defined as

$$[v_i, v_j] = v_iv_j - v_jv_i \qquad i,j = 1,\ldots\ldots,5.$$

Table 1. Commutator table for 5 parameter Lie algebra

| $[v_i, v_j]$ | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ |
|---|---|---|---|---|---|
| $v_1$ | 0 | 0 | $v_2$ | $v_1$ | $2v_1$ |
| $v_2$ | 0 | 0 | 0 | $\frac{1}{2}v_2$ | $v_3$ |
| $v_3$ | $-v_2$ | 0 | 0 | $-\frac{1}{2}v_3$ | 0 |
| $v_4$ | $-v_1$ | $-\frac{1}{2}v_2$ | $-\frac{1}{2}v_3$ | 0 | $v_5$ |
| $v_5$ | $-2v_4$ | $-v_3$ | 0 | $-v_5$ | 0 |

Since the set of invariant vector fields forms a linear space any linear combination of these vectors is also an invariant vector field. In section IV we determine solutions of equation (2) by using invariant vector fields $v_5$ and $v_1 + v_5$.

*IV. Analytical solution:*

In this section we determine some solutions of equation (2). To determine the analytic solution of equation (2) vector field $v_5$ is used to define new independent and dependent variables. Correspond to vector field $v_5$ we get the characteristic equation

$$\frac{dt}{t^2} = \frac{dx}{tx} = \frac{dc}{\left(-tc - \frac{1}{2a}\,x - \frac{b}{2a}\,t\right)} = \frac{d\tau}{1}$$

From above characteristic equation define new variables $\tau = -\frac{1}{t}$, $\theta = \frac{x}{t}$ and

$\mu = \frac{b}{2a}\,t + \frac{1}{2a}\,x + tc$. We treat $\mu$ as new dependent variable. Since, $\tau$ and $\theta$ are functions of $x$ and $t$ we treat $\tau$ and $\theta$ as new independent variables. With this choice of new variables equation (2) becomes,

$$\mu_\tau = 2a\mu\mu_\theta + \beta\mu_{\theta\theta} \tag{6}$$

Equation (6) is a standard one-dimensional Burgers equation. Here, we use First integral method[1] to find analytical solution of equation (6). We use transformation

$\mu(\theta,\tau) = f(\eta)$, where $\eta = \theta - \beta\tau$ (7)

Then by using chain rule,

$$\frac{\partial}{\partial\tau}(.) = -\beta\frac{d}{d\eta}(.), \frac{\partial}{\partial\theta}(.) = \frac{d}{d\eta}(.), \frac{\partial^2}{\partial\theta^2}(.) = \frac{d^2}{d\eta^2}(.), \ldots\ldots \tag{8}$$

With the representation given in equation (7) equation (6) gets converted in to an ordinary differential equation

$$\frac{d^2f}{d\eta^2} = -\frac{df}{d\eta} - \frac{2a}{\beta}f\frac{df}{d\eta} \tag{9}$$

We define new independent variables as

$$X(\eta) = f(\eta), Y(\eta) = f_\eta(\eta) \tag{10}$$

this leads to a system of ordinary differential equations,

$$X_\eta = Y$$

and

$$Y_\eta = -Y - \frac{2a}{\beta}XY$$

In the phase plane we get

$$\frac{dY}{dX} = -\left(1 + \frac{2a}{\beta}\,X\right)$$

By integrating above equation, we get

$$Y = \frac{dX}{d\eta} = -\left(1 + \frac{a}{\beta}\,X\right)X + k_1,$$

where $k_1$ is integrating constant.

On integrating above differential equation, we get

$$X = -\frac{\beta}{2a} + K\coth\left[\frac{Ka}{\beta}(\eta - k_2)\right],$$

where $k_2$ is integrating constant and

$$K = \frac{1}{2}\sqrt{\frac{\beta^2}{a^2} + 4\frac{k_1\beta}{a}}.$$

But $X = \mu$ and $c = -\frac{b}{2a} - \frac{x}{2at} + \frac{\mu}{t}$

therefore, we have

$$c = -\frac{b}{2a} - \frac{1}{t}\left\{\frac{x+\beta}{2a} - K\coth\left[\left(\frac{Ka}{\beta}\right)\left(\frac{x+\beta}{t} - k_2\right)\right]\right\} \tag{11}$$

which is analytical solution of equation (2).

The solution (11) obtained from vector field $v_5$ is defined for $t > 0$. Non-singular solution of equation (2) is obtained from vector field $v_1+v_5$. The characteristic equation corresponding to the vector field $v_1+v_5$ is

$$\frac{dt}{1+t^2} = \frac{dx}{tx} = \frac{dc}{\left(-tc - \frac{1}{2a}\,x - \frac{b}{2a}\,t\right)} = \frac{d\tau}{1}$$

From the above characteristic equation define new independent variables $\tau$ and $\theta$ as $\tau = \tan^{-1} t$ and $\theta = \frac{x}{\sqrt{1+t^2}}$. And the new dependent variable denoted by $\mu$ is defined as

$$\mu = t + \frac{(1+t^2)(2ac+b)}{x}$$

with this choice of new variables, equation (2) becomes

$$\mu_\tau = 1 + \mu(\mu + \theta\,\mu_\theta) + \frac{\beta}{\theta}(2\mu_\theta + \theta\,\mu_{\theta\theta}) \qquad (12)$$

Equation (12) is purely non-linear and different functional forms will provides different solutions to equation (12). In this section we report two different solutions.

Suppose, $\mu + \theta\,\mu_\theta = k(\tau)$

then $\mu = k(\tau) + \frac{1}{\theta} k_1(\tau)$ (13)

Since $(\mu + \theta\,\mu_\theta)_\theta = 0$, equation (12) becomes

$$\mu_\tau = 1 + \mu k(\tau) \qquad (14)$$

From equation (13) and (14) we get,

$$k'(\tau) + \frac{1}{\theta} k_1'(\tau)$$

$$= 1 + \left[k(\tau) + \frac{1}{\theta}\,k_1(\tau)\right] k(\tau)$$

On equating the coefficients of equal powers of $\theta$ we get $k(\tau)$ and $k_1(\tau)$ as follows

$$k(\tau) = \tan(\tau + c_1) \text{ and}$$

$$k_1(\tau) = c_2 \sec(\tau + c_1)$$

Where, $c_1$ and $c_2$ are constants of integration.

Thus from equation (13) we get

$$\mu = \tan(\tau + c_1) + \frac{1}{\theta} c_2 \sec(\tau + c_1)$$

Since $c = \frac{1}{2a}\left[\frac{x}{1+t^2}(\mu - t)\right] - \frac{b}{2a}$ we have

$$c = \frac{-b}{2a} + \frac{c_2 + x\sin c_1}{2a(\cos c_1 - t\,\sin c_1)} \qquad (15)$$

Another solution of equation (12) can be obtained by assuming that

$$\mu = \frac{f_1(\tau)}{\theta^2} + f_2(\tau) \qquad (16)$$

Since equation (16) is solution of equation (12), we get

$$\frac{f_1'(\tau)}{\theta^2} + f_2'(\tau) = 1 + f_2^2(\tau) - \frac{f_1^2(\tau)}{\theta^4} + 2\beta\,\frac{f_1(\tau)}{\theta^4}$$

On equating the coefficients of equal powers of $\theta$ we get $f_1(\tau)$ and $f_2(\tau)$ as follows

$$f_1(\tau) = 2\beta \text{ and } f_2(\tau) = \tan(\tau + c_3)$$

where $c_3$ is constant of integration.

Thus from equation (16) we get

$$\mu = \frac{2\beta}{\theta^2} + \tan(\tau + c_1)$$

Since $c = \frac{1}{2a}\left[\frac{x}{1+t^2}(\mu - t)\right] - \frac{b}{2a}$ we have

$$c = \frac{1}{2a}\left(\frac{2\beta}{x} + \frac{x\,\tan c_1}{1 - t\,\tan c_1}\right) - \frac{b}{2a}.$$

*V. One dimensional optimal system:*

In section IV we have seen that there are many solutions available for equation (2) and it necessary to classify all solutions into different categories. This classification is achieved by constructing one dimensional optimal system. The problem of finding an optimal system of subgroups is equivalent to that of finding an optimal system of subalgebras. For one-dimensional subalgebra this classification is essentially the same as the problem of classifying the orbits of the adjoint representation. From the commutator table we obtain the adjoint representation table 2. To compute adjoint representation we use the Lie series

$$Ad(exp(\varepsilon w))w_0 = w_0 - \varepsilon[w, w_0] + \frac{\varepsilon^2}{2}[w,[w,w_0]] - \cdots$$

Table 2. Adjoint representation of Lie subalgebra of dimension 5

| $Ad$ | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ |
|---|---|---|---|---|---|
| $v_1$ | $v_1$ | $v_2$ | $v_3 -\varepsilon v_2$ | $v_4 -\varepsilon v_1$ | $v_5 -2\varepsilon v_4 +\varepsilon^2 v_1$ |
| $v_2$ | $v_1$ | $v_2$ | $v_3$ | $v_4 -\frac{1}{2}\varepsilon v_2$ | $v_5 - \varepsilon v_3$ |
| $v_3$ | $v_1 +\varepsilon v_2$ | $v_2$ | $v_3$ | $v_4 + \frac{1}{2}\varepsilon v_3$ | $v_5$ |
| $v_4$ | $\varepsilon^2 v_1$ | $e^{\frac{1}{2}\varepsilon} v_2$ | $e^{-\frac{1}{2}\varepsilon} v_3$ | $v_4$ | $e^{-\varepsilon} v_5$ |
| $v_5$ | $v_1 + 2\varepsilon v_4 + \varepsilon^2 v_5$ | $v_2 +\varepsilon v_3$ | $v_3$ | $v_4 +\varepsilon v_5$ | $v_5$ |

For finding one-dimensional optimal system, we consider

$$v = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 + a_5v_5$$

Suppose $a_5 \neq 0$ without loss of generality let $a_5 = 1$. If we act on such a $v$ by $Ad(exp(a_3v_2))$, then we can make the coefficient of $v_3$ vanish and we get

$$v' = Ad(exp(a_3v_2))v$$
$$= a_1'v_1 + a_2'v_2 + a_4'v_4 + v_5$$

The coefficients $a_1', a_2', a_4'$ are functions of $a_1, a_2, a_3, a_4$. Next we act on $v'$ by $Ad\left(exp\left(\frac{1}{2}a_4'v_1\right)\right)$ to cancel the coefficient of $v_4$, leading to

$$v'' = Ad\left(exp\left(\frac{1}{2}a_4'v_1\right)\right)v'$$
$$= a_1''v_1 + a_2''v_2 + v_5$$

For certain $a_1'', a_2''$ depending on $a_1', a_2', a_4'$.

Next we act on $v''$ by $Ad\left(exp\left(\frac{-a_2''}{a_1''}v_3\right)\right)$ to eliminate the coefficient $v_2$, leading to

$$v''' = Ad\left(exp\left(\frac{-a_2''}{a_1''}v_3\right)\right)v'' = a_1'''v_1 + v_5$$

For certain $a_1'''$ depending on $a_1'', a_2''$. we can further act on $v'''$ by $d(exp(\varepsilon v_4))$. This has the net effect of scaling the coefficients of $v_1$ and $v_5$

$$v'''' = Ad(exp(\varepsilon v_4)) = a_1'''e^{2\varepsilon}v_1 + v_5$$

We can make the coefficient of $v_1$ either or +1,–1 or 0. Then one-dimensional subalgebras spanned by a general vector field $v$ with $a_5 \neq 0$ is equivalent to an algebra spanned by $\{v_1+v_5, -v_1 + v_1 + v_5, + v_5\}$.

The remaining one-dimensional subalgebras are spanned by vectors of the form

$$v = a_1 v_1 + a_2 v_2 + a_3 v_3 + v_4$$

If we act on such a $v$ by Ad (exp $(a_1 v_1)$), we can make the coefficient of $v_1$ vanish and we get

$$v' = Ad\big(exp(a_1 v_1)\big) = a_2' v_2 + a_3' v_3 + v_4$$

For certain $a_2', a_3'$ depending on $a_1, a_2, a_3$. Next we act on $v'$ by $Ad\left(exp\left(2a_2' v_2\right)\right)$ to cancel the coefficient $v_2$, leading to

$$v'' = Ad\left(exp\left(2a_2' v_2\right)\right) v' = a_3'' v_3 + v_4$$

For certain $a_3''$ depending on $a_{2,}', a_3'$. Next we act on $v''$ by $Ad\left(exp\left(-2a_3'' v_3\right)\right)$ to cancel the coefficient $v_3$, leading to

$$v''' = Ad\left(exp\left(-2a_3'' v_3\right)\right) v'' = v_4$$

Every one-dimensional subalgebra generated by $v$ with $a_5 = 0, a_4 \neq 0$ is equivalent to the algebra spanned by $\{v_4\}$.

Similarly with $a_5 = a_4 = 0, a_3 \neq 0$ and $a_3=1$, we get

$$v = a_1 v_1 + a_2 v_2 + a_3$$

If we act on such a $v$ by $Ad\big(exp(a_2 v_1)\big)$, we can make the coefficient of $v_2$ vanish

$$v' = Ad\big(exp(a_2 v_1)\big) v = a_1' v_1 + v_3$$

For certain $a_1'$ depending on $a_1, a_2$. We can further act on $v'$ by $Ad\big(exp(\varepsilon v_4)\big)$. this operation has the net effect of scaling the coefficients of $v_1$ and $v_3$

$$v'' = Ad\big(exp(\varepsilon v_4)\big) v' = v_1' e^{\frac{1}{2}\varepsilon} v_1 + v_3$$

We can make the coefficient of $v_1$ either $+1, -1$ or $0$. Then one-dimensional subalgebras generated by $v$ with $a_5 = a_4 = 0$, $a_3 \neq 0$ is equivalent to an algebra spanned by $\{v_1+v_3, -v_1+v_3, v_3\}$.

The remaining one-dimensional subalgebras are spanned by vectors of the above form with $a_5 = a_4 = a_3 = 0$, $a_2 \neq 0$ and $a_2 = 1$, $v$ becomes,

$$v = a_1 v_1 + a_2$$

If we act on such a $v$ by $Ad\left(exp\left(-\frac{1}{a_1} v_3\right)\right)$ then the coefficient of $v_2$ vanish:

$$v' = Ad\left(exp\left(-\frac{1}{a_1} v_3\right)\right) v = a_1' v_1$$

Every one-dimensional subalgebra generated by $v$ with $a_5 = a_4 = a_3 = 0$, $a_2 \neq 0$ is equivalent to the algebra spanned by $\{v_1\}$.

Thus the optimal system of one-dimensional subalgebras are spanned by

$$(a_1) \qquad v_5 = t^2 \frac{\partial}{\partial t} + tx \frac{\partial}{\partial x} + \left(-tc - \frac{1}{2a} x - \frac{b}{2a} t\right) \frac{\partial}{\partial c}$$

$$(a_2) \quad v_1 + v_5 = (1 + t^2) \frac{\partial}{\partial t} + tx \frac{\partial}{\partial x}$$

$$+\left(-tc-\frac{1}{2a}\,x-\frac{b}{2a}\,t\right)\frac{\partial}{\partial c}$$

$$(a_3)\ -v_1+v_5=(-1+t^2)\frac{\partial}{\partial t}+tx\frac{\partial}{\partial x}$$

$$+\left(-tc-\frac{1}{2a}\,x-\frac{b}{2a}\,t\right)\frac{\partial}{\partial c}$$

$$(b)\quad v_4=t\frac{\partial}{\partial t}+\frac{1}{2}\,x\frac{\partial}{\partial x}+\left(-\frac{1}{2}c-\frac{b}{4a}\right)\frac{\partial}{\partial c}$$

$$(c_1)\quad v_3=t\frac{\partial}{\partial x}-\frac{1}{2a}\frac{\partial}{\partial c}$$

$$(c_2)\quad v_1+v_3=\frac{\partial}{\partial t}+t\frac{\partial}{\partial x}-\frac{1}{2a}\frac{\partial}{\partial c}$$

$$(c_3)\quad -v_1+v_3=-\frac{\partial}{\partial t}+t\frac{\partial}{\partial x}-\frac{1}{2a}\frac{\partial}{\partial c}$$

$$(d)\quad v_1=\frac{\partial}{\partial t}$$

## Conclusion

The symmetry algebra of Burger-like equation (2) is spanned by five vector fields. The solutions of equation (2) are translation invariant in time as well as in space co-ordinates. A singular solution of equation (2) is obtained from vector field. Two solutions are obtained by using a linear combination of two vector fields and first integral method. Optimal system consisting of eight one-dimensional subalgebras is obtained. The generators of there subalgebras are listed at the end.

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