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On The Mixed Boundary Value Problem Of Steady State Heat Conduction In A Domain Formed By Cementing Two Plano-Convex Solids

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Abstract

The contact problem of two conducting plano-convex solids having different conductivities is considered assuming that steady state heat conduction takes place. The problem is formulated so as to involve a pair of dual integral equations having Legendre functions with complex index. These equations are reduced to a single integral equation which is then solved iteratively. Lastly, the quantities of physical interest are found out.

Key words: 45A05 Linear integral equation, 45B05 Fredholm integral equation, 45F10 Dual integral equations, 33- XX Legendre function, 42C05 Orthogonal functions and polynomials, 44A15 Special Transforms (Hankel Transform, Mehler- Fook Transform), 44A20 Transforms of special functions, 34BXX Boundary value problem, 32A05 Power Series, 74F05 Thermal effect, 80A20 Heat flow, 35J05 Laplace equation.

Introduction

In recent years several papers have been published on the dual integral equations. These are important while solving the boundary value problems of Mathematical Physics with mixed boundary conditions. Majority of them have been considered through the Hankel transformation whose kernels are expressible in cylindrical functions. Also the dual integral equations with kernels expressible in Legendre functions with complex index have recently been investigated[1,4,5,7]. These equations belong to the class connected with Mehler-Fook integral transformation and are of considerable interest in various problems of Mathematical Physics. References of mixed boundary value and boundary value problems of heat conduction are available[2,6,8].

In the present paper we have reduced our problem into simultaneous dual integral equations having Legendre functions with complex index, and then they are reduced to Fredholm integral equation of second kind. Finally it is solved iteratively.

Formulation of the problem :

To solve the problem we introduce a system of toroidal coordinates (α,β) related to cylindrical coordinates (r,z) by the expressions.

$$r=\frac{a\sinh\alpha}{\cosh\alpha+\cos\beta}$$

$$0<\alpha<\infty, 0<\beta<2\pi.$$

$$z=\frac{a\sin\beta}{\cosh\alpha+\cos\beta}$$

The temperature distribution in two plano-convex solids ABCOA & ADCOA is here considered. The total region ABCDA is formed by two intersecting spheres as shown in the Fig. 1 EC and FA portion of cemented surface of the solids are perfectly insulated and A and C are rigidly connected. In the upper solid the temperature function is prescribed along ABC. Hence we have the boundary conditions[3]:

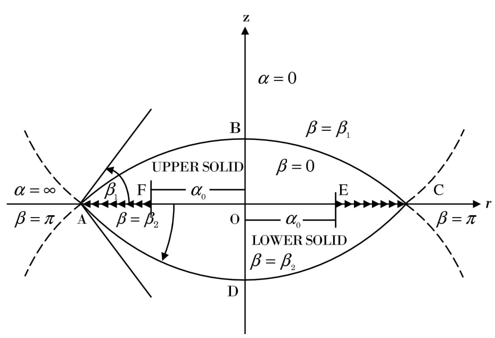

Fig. 1 – Region ABCDA formed by two intersecting spheres (the insulation takes place along the shaded lines).

$$u_1=v_1(\alpha),\quad \beta=\beta_1,\quad 0<\alpha<\infty, \tag{1}$$

$$\frac{\partial u_1}{\partial\beta}=0,\quad \beta=0,\ \alpha_0<\alpha<\infty. \tag{2}$$

The cemented portion denoted between EF is perfectly conducting. Since we shall assume that the surface ADC of the lower solid, is a sink, at the surface of separation of two media we have the following boundary conditions.

$$u_1=u_2,\quad \beta=0,\quad 0<\alpha<\alpha_0, \tag{3}$$

$$K_1\frac{\partial u_1}{\partial\beta}=K_2\frac{\partial u_2}{\partial\beta},\ \beta=0,\ 0<\alpha<\alpha_0, \tag{4}$$

where K_1 and K_2 are the conductivities of the upper and lower solids respectively. As already mentioned the cemented surface has EC and FA part insulated and on the lower surface ADC temperature is taken to be zero. On these lines if we take $\beta=-\beta_2$ on the lower surface, we can write:

$$\frac{\partial u_2}{\partial\beta}=0,\quad \beta=0,\quad \alpha_0<\alpha<\infty, \tag{5}$$

$$u_2=0,\quad \beta=-\beta_2\quad 0<\alpha<\infty, \tag{6}$$

where u_1 and u_2 are the solutions of Laplace's equation

$$\left.\begin{aligned}\nabla^2u_1&=0, &(a)\\ \nabla^2u_2&=0, &(b)\end{aligned}\right\} \tag{7}$$

For the upper plano-convex solid we assume the solution of the Laplace's equation in the from:

$$u_1 = v_1(\alpha) + \sqrt{\cosh\alpha + \cos\beta}\int_0^\infty A(r)\frac{\sinh(\beta_1 - \beta)r}{\cosh\beta_1 r}\tanh\pi r\,\rho_{-\frac{1}{2}+ir}(\cosh\alpha)\,dr,$$

$$0 < \beta < \beta_1, \quad 0 < \alpha < \infty \qquad (8)$$

This form satisfies the condition (1). Also $A(\tau)$, is unknown constant. For the lower solid a suitable temperature function is

$$u_2 = \sqrt{\cosh\alpha + \cos\beta}\int_0^\infty \frac{\sinh(\beta + \beta_2)r}{\cosh\beta_2 r}B(r)\tanh\pi r\,\rho_{-\frac{1}{2}+ir}(\cosh\alpha)\,dr,$$

$$-\beta_2 < \beta < 0, \quad 0 < \alpha < \infty \qquad (9)$$

Here $B(r)$ is unknown constant. This form satisfies the conditions (6). In satisfying the boundary conditions (3), (4), (2) & (5) the following equations are obtained :

$$\int_0^\infty \left[B(r)\tanh\beta_2 r - A(r)\tanh\beta_1 r\right]\tanh\pi r\,\rho_{-\frac{1}{2}+ir}(\cosh\alpha)\,dr = \frac{v_1(\alpha)}{\sqrt{1+\cosh\alpha}},$$

$$0 < \alpha < \alpha_0, \qquad (10)$$

$$\int_0^\infty \left[B(r) - \sigma A(r)\right] r\tanh\pi r\,\rho_{-\frac{1}{2}+i\tau}(\cosh\alpha)\,dr = 0,$$

$$0 < \alpha < \alpha_0, \qquad (11)$$

$$\int_0^\infty r A(r)\tanh\pi r\,\rho_{-\frac{1}{2}+i\tau}(\cosh\alpha)\,dr = 0,$$

$$\alpha_0 < \alpha, \qquad (12)$$

$$\int_0^\infty r B(r)\tanh\pi r\,\rho_{-\frac{1}{2}+i\tau}(\cosh\alpha)\,dr = 0,$$

$$\alpha_0 < \alpha, \qquad (13)$$

where $\sigma = \dfrac{K_1}{K_2}$

Some useful Results :

We give below some results[1], which we shall now make use of.

$$\int_0^\infty \cos rt\,\rho_{-\frac{1}{2}+i\tau}(\cosh\alpha)\,dr = \left[2(\cosh\chi - \cosh t\right]^{-\frac{1}{2}} H(\alpha - t), \qquad (14)$$

$$\int_0^\infty \rho_{-\frac{1}{2}+i\tau}(\cosh\alpha)\tanh\pi r\sin rs\,dr$$

$$= \left[2\;(\cosh s - \cosh\alpha\right]^{-\frac{1}{2}} H(s - \alpha), \qquad (15)$$

Here $H(t)$ is Heavy-side unit function.

The Mehler-Fock transform[1] is given by

$$f(\alpha) = \int_0^\infty g(r)\, \rho_{-\frac{1}{2}+i\tau}(\cosh\alpha)\, dr, \tag{16}$$

then

$$g(r) = r \tanh \pi r \int_0^\infty f(\alpha)\, \rho_{-\frac{1}{2}+i\tau}(\cosh\alpha)\, d\alpha, \tag{17}$$

and hence the following relation can easily be derived :

$$\cos r s = \frac{1}{\sqrt{2}} \frac{d}{ds} \int_0^s \frac{\rho_{-\frac{1}{2}+i\tau}(\cosh\alpha)\sinh\alpha\, d\alpha}{(\cosh s - \cosh\alpha)^{\frac{1}{2}}} \tag{18}$$

Solution Of Simultaneous Dual Integral Equations Involving Legendre Functions Of Imaginary Argument :

We shall now solve the equations (10) to (13). Let us assume

$$A(r) = \int_0^{\alpha_0} \phi(t) \cos rt\, dt, \tag{19}$$

where $\phi(t)$ is unknown. Equation (19) can be written in the form after integrating it by parts.

$$A(r) = \frac{\phi(\alpha_0)\sin r\alpha_0}{r} - \frac{1}{r}\int_0^{\alpha_0} \phi'(t) \sin r t\, dt. \tag{20}$$

With the help of (20) and then (15) it can be shown that (12) is satisfied identically for any function $\phi(t)$ which has a continuous derivative. We also have

$$\int_0^\infty r A(r) \tanh \pi r\, \rho_{-\frac{1}{2}+i\tau}(\cosh\alpha)\, dr = \frac{\phi(\alpha_0)}{\sqrt{2(\cosh\alpha_0 - \cosh\alpha)}} - \frac{1}{\sqrt{2}} \int_\alpha^{\alpha_0} \frac{\phi'(t)\, dt}{\sqrt{\cosh t - \cosh\alpha}},$$

$$0 < \alpha < \alpha_0 \tag{21}$$

Now from (11)

$$\int_0^\infty r B(r) \tanh \pi r\, \rho_{-\frac{1}{2}+i\tau}(\cosh\alpha)\, dr = \frac{\sigma\,\phi(\alpha_0)}{\sqrt{2(\cosh\alpha_0 - \cosh\alpha)}} - \frac{\sigma}{\sqrt{2}} \int_\alpha^{\alpha_0} \frac{\phi'(t)\, dt}{\sqrt{\cosh t - \cosh\alpha}},$$

$$0 < \alpha < \alpha_0 \tag{22}$$

Making use of (16) & (17) we can get easily from (13) & (22)

$$B(r) = \frac{\sigma}{\sqrt{2}} \quad \phi(\alpha_0) \int_0^{\alpha_0} \frac{\sinh\alpha \, \rho_{-\frac{1}{2}+i\tau}(\cosh\alpha)\, d\alpha}{\sqrt{\cosh\alpha_0 - \cosh\alpha}} - \frac{\sigma}{\sqrt{2}} \int_0^{\alpha_0} \rho_{-\frac{1}{2}+i\tau}(\cosh\alpha) \sinh\alpha \, d\alpha,$$

$$\int_\alpha^{\alpha_0} \frac{\phi'(t)dt}{\sqrt{\cosh t - \cosh\alpha}} \tag{23}$$

On interchanging the order of integrations in the second integral of (23) and then integrating by parts and finally using (18), we get :

$$B(r) = \sigma \int_0^{\alpha_0} \cos r\, t \, \phi(t)\, dt. \tag{24}$$

Equation (10) can be written in the from

$$\int_0^\infty B(r)\left[\tanh\beta_2\, r \tanh\pi\, r - 1\right] \rho_{-\frac{1}{2}+i\tau}(\cosh\alpha)\, dr$$

$$+ \int_0^\infty A(r)\left[1 - \tanh\beta_1\, r \tanh\pi\, r\right] \rho_{-\frac{1}{2}+i\tau}(\cosh\alpha)\, dr$$

$$+ \int_0^\infty B(r)\, \rho_{-\frac{1}{2}+i\tau}(\cosh\alpha)\, dr - \int_0^\infty A(r)\, \rho_{-\frac{1}{2}+i\tau}(\cosh\alpha)\, dr = \frac{v_1(\alpha)}{\sqrt{1+\cosh\alpha}}$$

$$0 < \alpha < \alpha_0 \tag{25}$$

If we substitute the values of $B(r)$ & $A(r)$ from (24) & (19) in (25), then using (14) we find that

$$(\sigma - 1) \int_0^\alpha \frac{\phi(t)\, dt}{\sqrt{\cosh\alpha - \cosh t}} = \frac{\sqrt{2}\, v_1(\alpha)}{\sqrt{1+\cosh\alpha}} - \sqrt{2} \int_0^{\alpha_0} \phi(t)\, dt \int_0^\infty \cosh(\pi - \beta_1)\, r.$$

$$.\frac{\cos r\, i\, \rho_{-\frac{1}{2}+i\tau}(\cosh\alpha)\, dr}{\cosh\beta_1 r \cosh\pi\, r} + \sigma\sqrt{2} \int_0^{\alpha_0} \phi(t)\, dt.$$

$$.\int_0^\infty \frac{\cos(\pi - \beta_2)\, r \cos r\, t}{\cosh\beta_2 r \cosh\pi\, r} \, \rho_{-\frac{1}{2}+i\tau}(\cosh\alpha)\, dr,$$

$$0 < \alpha < \alpha_0 \tag{26}$$

Equation (26) is Abel type. Hence the solution is obtained by using (18) :

$$\phi(t) = \frac{\sqrt{2}}{\pi(\sigma-1)}\frac{d}{dt}\int_0^t \frac{\sinh\alpha\, v_1(\alpha)\, d\alpha}{\sqrt{1+\cosh\alpha}\sqrt{\cosh t-\cosh\alpha}} - \frac{2}{\pi(\sigma-1)}$$

$$\int_0^{\alpha_0}\phi(u)\,du \int_0^{\infty}\frac{\cosh(\pi-\beta_1)\,r}{\cosh\beta_1 r\cosh\pi\, r}\cos r\, t\cos r\, u\, dr +$$

$$+\frac{2\sigma}{\pi\,(\sigma-1)}\int_0^{\alpha_0}\phi(u)\,du\int_0^{\infty}\frac{\cosh(\pi-\beta_2)\,r}{\cosh\beta_2 r\cosh\pi\, r}\cos r\, t\cos r\, u\, dr$$

$$0 < t < \alpha_0 \tag{27}$$

If $v_1(\alpha) = \frac{1}{\sqrt{2}}$, (constant),

then (27) can be written in the form

$$\phi(t) = \frac{\sec h\left(\frac{t}{2}\right)}{(\sigma-1)\,\pi} + \frac{2}{(\sigma-1)\,\pi}\int_0^{\alpha_0}\phi(u)\left[\, K_1(u,t) + K_2(u,t)\,\right]du\,,$$

$$0 < t < \alpha_0 \tag{28}$$

where

$$\left.\begin{aligned} K_1\,(u,t) &= -\int_0^{\infty}\frac{\cosh\,(\pi-\beta_1)\,r}{\cosh r\,\beta_1\cosh\pi\, r}\cos r\, t\cos r\, u\, dr, && (a)\\ &\text{and}\\ K_2\,(u,t) &= \sigma\int_0^{\infty}\frac{\cosh\,(\pi-\beta_2)\,r}{\cosh r\,\beta_2\cosh\pi\, r}\cos r\, t\cos r\, u\, dr, && (b)\end{aligned}\right\} \tag{29}$$

Equation (28) is Fredholm integral equation of second kind having kernel $K_1\,(u,t) + K_2\,(u,t)$. Equation (28) is a standard equation.

Solution of fredholm integral equation :

Equation (28) can be solved for any suitable particular value of β_1 and β_2. Here we shall get the iterative solution of the Fredholm integral equation and obtain the solution of (28) as a power series in α_0 provided that α_0 is sufficiently small.

If $\beta_1 = \frac{\pi}{2}$ & $\beta_2 = \frac{\pi}{2}$ then the domain ABCDA represents a sphere. Equation (28) reduces to

$$\phi(t)=\frac{\sec h\left(\frac{t}{2}\right)}{\pi(\sigma-1)}+\frac{2}{\pi}\int_0^{\alpha_0}\frac{\phi(u)\cosh\frac{u}{2}\cosh\frac{t}{2}\,du}{\cosh u+\cosh t},$$

$$0<t<\alpha_0 \qquad (30)$$

If we take $t=r\alpha_0$, $u=x\,\alpha_0$,

$\phi(r\,\alpha_0)=\psi(r\,\alpha_0)=\psi(r)$; say then (30) takes the form

$$\psi(r)=\frac{\sec h\frac{r\,\alpha_0}{2}}{\pi(\sigma-1)}+\frac{2\alpha_0}{\pi}\int_0^1\psi(x)\frac{\cosh\frac{r\,\alpha_0}{2}\cosh\frac{x\,\alpha_0}{2}}{\cosh r\,\alpha_0+\cosh x\,\alpha_0}dx,$$

$$0<r<1 \qquad (31)$$

If α_0 is very small such that $\alpha_0<<1$, then we can represent

$$\frac{\cosh\frac{r\,\alpha_0}{2}\cosh\frac{x\,\alpha_0}{2}}{\cosh r\,\alpha_0+\cosh x\,\alpha_0}=\frac{1}{2}-\frac{\alpha_0^{\;2}}{16}(x^2+r^2)$$

$$+\frac{5\,\alpha_0^{\;4}}{4!\,32}(x^4+r^4+6x^2r^2)+o(\alpha_0^{\;6})+...... \qquad (32)$$

and

$$\sec h\frac{r\,\alpha_0}{2}=1-\frac{r^2\,\alpha_0^{\;2}}{8}+\frac{5r^4\,\alpha_0^{\;4}}{384}+........,\alpha_0 r<\pi \qquad (33)$$

If we represent the solution of (32) in the form

$$\psi(r)=n_0(r)+\alpha_0\,n_1(r)+\alpha_0^{\;2}\,n_2(r)+\alpha_0^{\;3}\,n_3(r)+........ \qquad (34)$$

Then by substituting the value of $\psi(r)$ in (31) and equating the like powers of α_0 we obtain:

$$n_0(r)=\frac{1}{(\sigma-1)\pi}$$

$$n_1(r)=\frac{1}{(\sigma-1)\pi^2}$$

$$n_2(r)=\frac{-r^2}{8\pi(\sigma-1)}+\frac{1}{\pi}\int_0^1 n_1(x)\,dx=\frac{-r^2}{8\pi(\sigma-1)}+\frac{1}{\pi^3(\sigma-1)}$$

$$n_3(r)=-\frac{1}{8\pi}\int_0^1(x^2+r^2)\,n_0(x)\,dx+\frac{1}{\pi}\int_0^1 n_2(x)\,dx$$

$$n_3(r)=-\frac{\left(r^2+\frac{1}{8}\right)}{8\pi^2(\sigma-1)}-\frac{1}{24(\sigma-1)\pi^2}+\frac{1}{\pi^4(\sigma-1)}$$

$$n_4(r)=-\frac{5r^4}{384(\sigma-1)\pi}+\frac{5}{4!16\pi}\int_0^1 n_0(x)(x^4+r^4+6x^2r^2)\,dx-$$

$$-\frac{1}{8\pi}\int_0^1 n_1(x)(x^2+r^2)\,dx+\frac{1}{\pi}\int_0^1 n_3(x)\,dx$$

$$n_4(r) = \frac{5r^4}{384\,(\sigma-1)\,\pi} + \frac{5\left(r^4 + 2r^2 + \frac{1}{5}\right)}{4!16\,\pi^2\,(\sigma-1)} - \frac{\left(r^2 + \frac{1}{3}\right)}{8\pi^3\,(\sigma-1)} + \frac{1}{\pi^5\,(\sigma-1)}$$

$$-\frac{1}{8\pi^3\,(\sigma-1)}$$

Now from (34) we have

$$\psi\,(r) = \frac{1}{(\sigma-1)\,\pi} + \frac{\alpha_0}{\pi^2\,(\sigma-1)} + {\alpha_0}^2\left[\frac{1}{\pi^2\,(\sigma-1)} - \frac{r^2}{8\pi\,(\sigma-1)}\right]$$

$$+{\alpha_0}^3\left[\frac{1}{\pi^4\,(\sigma-1)} - \frac{1}{24\,(\sigma-1)\,\pi^2} - \frac{\left(r^2+\frac{1}{3}\right)}{8\pi^2\,(\sigma-1)}\right]$$

$$+{\alpha_0}^4\left[\frac{5\,r^4}{384\,(\sigma-1)\,\pi} + \frac{5\left(r^4 + 2\,r^2 + \frac{1}{5}\right)}{4!\;16\,\pi^2\,(\sigma-1)} - \frac{\left(r^2+\frac{1}{3}\right)}{8\,\pi^3\,(\sigma-1)}\right.$$

$$\left.-\frac{1}{8\pi^3\,(\sigma-1)} + \frac{1}{\pi^5\,(\sigma-1)}\right] + o({\alpha_0}^5) + \ldots\ldots\ldots \tag{35}$$

Equation (19) can be written in the form

$$A\,(r) = \alpha_0 \int_0^1 \psi\,(\tau) \cos \tau\, r \alpha_0\; dr, \tag{36}$$

Hence

$$A\,(r) = \frac{\alpha_0}{\pi\,(\sigma-1)} + \frac{{\alpha_0}^2}{\pi^2\,(\sigma-1)} + {\alpha_0}^3\left[\frac{1}{\pi^3\,(\sigma-1)}\right.$$

$$\left.-\frac{1}{24\pi\,(\sigma-1)} - \frac{r^2}{6\pi\,(\sigma-1)}\right] + {\alpha_0}^4\left[\frac{-r^2}{6\pi^2\,(\sigma-1)} + \frac{1}{\pi^4\,(\sigma-1)}\right.$$

$$\left.-\frac{1}{8\,(\sigma-1)\,\pi^2}\right] + o\,({\alpha_0}^5) + \ldots\ldots\ldots \tag{37}$$

Some approximate Results :

We shall get the total quantity of heat passing per second through the circle of radius this circle being situated at the surface of separation of two media from upper solid to lower solid. This quantity of heat equal to

$$Q_1 = -2\pi k_1 \alpha_0 \int_0^{\alpha_0} \left(\frac{\partial u_1}{\partial \beta}\right)_{\beta=0} d\alpha$$

$$Q_1 = 2\sqrt{2}\,\pi k_1 \alpha_0 \int_0^{\alpha_0} \cosh\frac{\alpha}{2} d\alpha \int_0^{\infty} r A(r)\, \rho_{-\frac{1}{2}+i\tau}(\cosh\alpha) \tanh \pi r\, dr \qquad (38)$$

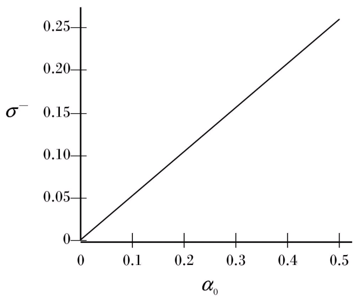

Fig. 2 – The variation of Q_1 with α_0 when the solids are silver (sterling) and lead.

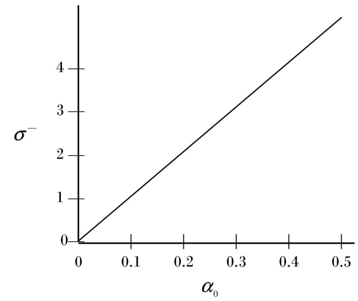

Fig. 3 – The variation of Q_1 with α_0 when the solids are silver (sterling) and copper.

Making use of (21) we find that

$$Q_1 = \sqrt{2}\,\pi k_1 \alpha_0 \left[\phi(\alpha_0) \int_0^{\alpha 0} \frac{\cosh\frac{\alpha}{2} d\alpha}{\sqrt{\sinh^2\frac{\alpha_0}{2} - \sinh^2\frac{\alpha}{2}}} \right.$$

$$\left. - \int_0^{\alpha_0} \cosh\frac{\alpha}{2} d\alpha \int_{\alpha}^{\alpha_0} \frac{\phi'(t)\, dt}{\sqrt{\sinh^2\frac{t}{2}\sinh^2\frac{\alpha}{2}}} \right] \qquad (39)$$

Interchanging the order of integration in the second term of (39) we obtain that

$$Q_1 = \sqrt{2}\,\pi k_1 \alpha_0 \left[\phi(\alpha_0) - \alpha_0 \int_0^1 \phi'(\alpha_0 r)\, dr \right]$$

Hence

$$Q_1 = \sqrt{2}\,\pi^2 k_1 \left[\frac{\alpha_0}{\pi(\sigma-1)} - \frac{\alpha_0^{\,2}}{\pi^2(\sigma-1)} \right.$$

$$+ \alpha_0^{\,3}\left(\frac{1}{\pi^3(\sigma-1)} - \frac{1}{8\pi(\sigma-1)}\right) + \alpha_0^{\,4}\left(\frac{1}{\pi^4(\sigma-1)}\right.$$

$$\left.\left. - \frac{5}{24\pi^2(\sigma-1)} + \frac{1}{8\pi(\sigma-1)}\right)\right] + o(\alpha_0^{\,5}) + \ldots\ldots\ldots \qquad (40)$$

The variation with α_0 of Q_1 is shown in Fig. 2 and 3. In Fig. 2 and 3, the set of solids are taken silver (sterling), lead and silver (sterling), copper respectively.

We shall now get the approximate results for temperature functions. For this purpose, we make use of (37). For the solid portion ABCO, if we take $v_1(\alpha) = \frac{1}{\sqrt{2}}$ and $\beta_1 = \frac{\pi}{2}$, equation (8) can be written as follows :

$$u_1 = \frac{1}{\sqrt{2}} + \sqrt{\cosh\alpha + \cos\beta}\,\Bigg[\left(\frac{\alpha_0\, I_1}{(\sigma-1)\,\pi} + \frac{\alpha_0^{\,2}\, I_1}{(\sigma-1)\,\pi}\right)$$

$$+\alpha_0^{\,3}\left(\frac{I_1}{\pi^3\,(\sigma-1)} - \frac{I_1}{24\,(\sigma-1)\,\pi} - \frac{I_2}{6\pi\,(\sigma-1)}\right)$$

$$+\alpha_0^{\,4}\left(\frac{I_1}{\pi^4\,(\sigma-1)} - \frac{I_1}{8\pi^2\,(\sigma-1)} - \frac{I_2}{6\pi^2\,(\sigma-1)}\right)\Bigg] + o\,(\alpha_0^{\,5}) + \dots\dots\dots$$

$$0 < \alpha < \infty, 0 < \beta < \frac{\pi}{2} \tag{41}$$

where

$$I_1 = \int_0^\infty \frac{\sinh\left(\frac{\pi}{2} - \beta\right) r}{\cosh\frac{\pi}{2} r}\; \rho_{-\frac{1}{2}+i\tau}\,(\cosh\alpha)\,\tanh\pi\, r\, dr, \tag{42}$$

and

$$I_2 = \int_0^\infty \frac{r^2\sinh\left(\frac{\pi}{2} - \beta\right) r}{\cosh\frac{\pi}{2} r}\; \rho_{-\frac{1}{2}+i\tau}\,(\cosh\alpha)\,\tanh\pi\, r\, dr. \tag{43}$$

(42) and (43) are convergent infinite integrals.

Similarly we can write the expression for temperature function assigned to the lower solid portion AOCDA as

$$u_2 = \sigma\sqrt{\cosh\alpha + \cos\beta}\,\Bigg[\frac{\alpha_0\, S_1}{(\sigma-1)\,\pi} + \frac{\alpha_0^{\,2}\, S_1}{(\sigma-1)\,\pi^2}$$

$$+\alpha_0^{\,3}\left(\frac{S_1}{\pi^3\,(\sigma-1)} - \frac{S_1}{24\,(\sigma-1)\,\pi} - \frac{S_2}{6\pi\,(\sigma-1)}\right)$$

$$+\alpha_0^{\,4}\left(\frac{S_1}{\pi^4\,(\sigma-1)} - \frac{S_1}{8\pi^2\,(\sigma-1)} - \frac{S_2}{6\pi^2\,(\sigma-1)}\right)\Bigg] + o\,(\alpha_0^{\,5}) + \dots\dots\dots$$

$$0 < \alpha < \infty, -\frac{\pi}{2} < \beta < 0 \tag{44}$$

where

$$S_1 = \int_0^\infty \frac{\sinh\left(\frac{\pi}{2}+\beta\right)r}{\cosh\frac{\pi}{2}r} \tanh \pi r \;\rho_{-\frac{1}{2}+i\tau}(\cosh\alpha)\,dr, \tag{45}$$

and

$$S_2 = \int_0^\infty \frac{r^2 \sinh\left(\frac{\pi}{2}+\beta\right)r}{\cosh\frac{\pi}{2}r} \tanh \pi r \;\rho_{-\frac{1}{2}+i\tau}(\cosh\alpha)\,dr. \tag{46}$$

Here (45) and (46) are again convergent infinite integrals. With the help of [3,6] the values of these integrals can be found out.

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