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Almost relative strongly M –flat modules

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Abstract

Throughout this paper we have defined a new concept of almost strongly M-flat modules[11] and we are motivated[10]. And given several properties of them[13]. This paper concept introduced as a generalization of flat modules.

There are two part in discussion

In first part we study Almost strongly M-flat modules and in second section we study about the homomorphism

$\theta_X : A\,\Sigma \underset{R}{\otimes} Hom_R\,(X^J, N^J) \rightarrow Hom_A\,(A\,\Sigma \underset{R}{\otimes} X^J\,,\ A\Sigma \underset{R}{\otimes} N^J\,)$

Where A is an R-algebra, X^J and N^J is R-modules.

And last result we prove M be almost strongly M-flat modules then following statement are equivalent.

a) Any finitely generated ideal I is almost M-projective
b) Every submodule of M is almost strongly M-flat modules.
c) Every flat module is almost strongly M-flat module.

Key words and phrases: Flat, module, sub- module, strongly injective and projective, strongly projective, annihilator, exact sequence, homomorphism, epimorphism, finitely generated.

Introduction

Definition 1.1: A module ΣU_R is flat relative to a module R^M (Or that Σ U is M-flat) in case the functor ($\Sigma U \underset{R}{\otimes} -$) Presserves the exactness of all short exact sequence with middle term M. Then of course, U is almost M-flat.

Definition 1.2: A ring R is said to be right strongly M-flat if every right R-module almost strongly M-flat[1-8].

Definition 1.3: Let K be any sub module of M is any indecomposable module.

Lemma 1.4: Let M=T $\otimes_R T^{-1}$ where T and T^{-1} are submodules of M. Given K is essential in M. Therefore K$\cap$ $T \neq 0$ *and* $K \cap T^{-1} \neq 0$. Since K is implies that K$\subset$T and K$\subset T^{-1}$. This implies that K$\subset$ $T\otimes_R T^{-1}$ which is contradiction.

Definition 1.5: T(M)={$^{U}/_{U}$ *is almost* $M - flat$}

$T^{-1}(U) = \{ ^{M}/_{U}$ *is almost* $M - flat\}$.

Definition 1.6: A module Σ U is called almost strongly M-flat if U∈T (M^J) for every indexing set J. Where $M^J = \Sigma\left(\prod_{i\in j} M_i\right)$.

Remark 1.7: Cleary U is almost strongly M-flat if and only if $M^J \in T^{-1}(U)$.

Lemma 1.8: If U is almost strongly M-flat ⇒ *U is M - flat.*

Proposition 1.9: Let U be M^J–faithful, then ΣU is almost strongly M-flat.

Proof: Consider any exact sequence $0 \to K \to \Sigma\ M^J$ of R-modules.

We show that $0 \to \Sigma\left(u \bigotimes k\right) \xrightarrow{U \bigotimes f} \Sigma\left(u \otimes M^J\right)$ is an exact sequence.

Let Σ ($u\otimes f$)($u\otimes k$)=0 where $u\in U$, $k\in K$. Then $u\otimes f$ (k)=0.
Hence by given condition f(k)=0 . Which gives k=0, since f is one-one

It follows that $u\otimes f$ is monomorphism[9-15].

Hence U is almost strongly M-flat.

Lemma 1.10: Let U, M be an R-modules. If U is strongly M-flat then U is flat over $\frac{R}{1_R}(M)$.

Proof: Suppose that Σ U is almost strongly M-flat.

Let $\overrightarrow{R} = \frac{R}{1_R}(M)$. Then by "Morita theorem" We have $\overrightarrow{R}$ can be embedded in a direct product of M, that is $\vec{R} \subseteq M^J$ for some index set J.

Due to fact that ΣU is M^J almost flat and $T^{-1}(U) = \{ ^{M}/_{U}$ *is* $M - flat\}$,

Is closed under sub module. We get, U is $\overrightarrow{R}$ -flat.

Lemma 1.11(Ozen): Let M be finitely presented and A-coherent Then M is quotient A-coherent.

Theorem 2.1: If U_S is almost strongly S^M–flat and Q is U-injective then $\vec{s}Hom_R(U,Q)$ is injective. Where

$\vec{s} = \frac{S}{1_S}(\mathrm{M})$ and $\mathrm{s} = \mathrm{End_R(u)}$.

Proof: $0 \to \Sigma\mathrm{I} \to \mathrm{f} \to \overrightarrow{S}$ be an exact sequence

U_S are almost strongly S^M–flat, hence by above lemma (1.11). Since is an exact sequence. Applying Hom_R(-,Q), we get the sequence $U_{\vec{S}}$ is flat

$0 \to U_{\vec{S}} \otimes \vec{I} \to \Sigma U \otimes f \to U_{\vec{S}} \otimes \vec{S} \cong \Sigma U$ is an exact sequence.

Applying Hom_R (-, Q), we get the sequence

$Hom_R(\Sigma U \otimes \vec{S}, Q) \to Hom_R(\Sigma U \otimes \vec{I}, Q) \to 0$ is exact consider the following commutative diagram.

$$\begin{array}{ccc} Hom_R(\Sigma U \otimes \vec{S}, Q) & Hom_R(\Sigma I \otimes f, Q) \to & Hom_R(\Sigma U \otimes \vec{I}, Q) \to 0 \\ \cong & & \cong \\ Hom_S(\vec{S}, \vec{s}Hom_R(\Sigma U, Q)) \to & Hom(f, Hom(\Sigma U, Q)) \to & Hom_S(\overline{\Sigma I}, \vec{s}Hom_R(\Sigma U, Q)) \end{array}$$

By the assumption, it is clear that Hom(ΣI $\otimes f$,Q) is natural epimorphism and hence Hom (f, Hom(ΣU,Q)) is also natural epimorphism.

Hence $\vec{s}Hom_R(\Sigma U, Q)$ is injective.

Proposition 2.2: Let ΣP, ΣM be an R-modules and S= End_R (M).

If M_S is injective and ΣP is M-projective then $Hom_R(\Sigma P, \Sigma M)_S$ is almost injective.

Proof: Let $O \to J \to \underline{\alpha} \to S$ be an exact sequence. Applying the functor Hom_S (-,M_S) and Hom_R(P,-) respectively, we get the sequence.

Hom_R (ΣP, $Hom_S(S,M)$) $\to$ Hom_R (ΣP, $Hom_S(J,M)$) $\to$ O is exact.

We show that Hom_S (- , $Hom_R(\Sigma P, \Sigma M)$) *is exact*. Let η be the canonical isomorphism. Consider the commutative diagram[21-23]

$$\begin{array}{ccc} Hom_S(S, Hom_R(\Sigma P, \Sigma M)) \to & Hom(\alpha, Hom(\Sigma P, \Sigma M)) \to & Hom_S(J, Hom_R(\Sigma P, \Sigma M)) \\ \cong \ \eta & & \cong \ \eta \\ Hom_R(\Sigma P, Hom_R(S, \Sigma M)) \to & Hom(P, Hom(\alpha, \Sigma M)) \to & Hom_R(P, Hom_R(J, \Sigma M)) \end{array}$$

By the assumption, Hom(ΣP, Hom(α, ΣM)) is natural epimorphism? This implies that the upper homomorphism Hom(α, Hom(ΣP, ΣM) is also epimorphism. Hence Hom_R (ΣP,ΣM)$_S$ is almost injective[16-20].

Definition 2.3: If R is commutative, and X , N are R-modules we will called the module Hom_R(N,X) the X – dual of N.

Let A be an R-algebraic, X and N are R-modules.

Then there exists a natural A-homorphism.

$\theta_X : A \, \Sigma \otimes_R Hom_R(X^J, N^J) \to Hom_A(A \, \Sigma \otimes_R X^J, A\Sigma \otimes_R N^J)$ is given be

$\theta_X : (a \otimes f)\{(b \otimes x)\} = ab \otimes \mathrm{f(x)}, \mathrm{a, b} \in A, x \in X, f \in Hom\,(X, N).$

We want to investigate conditions under which θ_R is an isomorphism for all R-module N. Clearly θ_X will be isomorphism provided A is flat R-algebra.

Theorem 2.4(Ozen): Let M be finitely presented , the following are equivalent

I. M is coherent .
II. (S)M-A I is closed.
III. (S)M-Af is closed under direct product under limit (where R is coherent ring.)

Theorem 2.5: If R is a Dedekind domain and M is flat R-modules, the $\emptyset \overset{limit}{\leftarrow} X_{n_j}$ is isomorphism if and only if ϕ Hom(- ,x) is an isomorphism.

Proof: "only if"

Suppose ϕ Hom (-, x) is an isomorphism ∀ X-duals. ⇒ ϕ Hom (-, x) is an isomorphism for X-dual of $\overset{limit}{\longrightarrow} Hom(R^{n_j}, X)$.

But Hom($\overset{limit}{\longrightarrow} Hom(R^{n_j}, X) \cong \overset{limit}{\longleftarrow} Hom(R^{n_j}, X)$

$\cong \overset{limit}{\longleftarrow} (Hom(R, X))^{n_j}$

$\cong \overset{limit}{\longleftarrow} X^{n_j}$

$\Rightarrow \emptyset\, Hom\left(\overset{limit}{\longrightarrow} R^{n_j}, X\right) = \emptyset \overset{limit}{\longleftarrow} X^{n_j}$ is an isomorphism.

Conversely: Suppose $\emptyset_{\overset{limit}{\longleftarrow} X^{n_j}}$ is an isomorphism. To show that $\emptyset_{Hom(-,X)}$ is an isomorphism ∀ X-duals. It suffices to consider the X-duals of flat modules.

Since if t is the torsion submodule of an arbitrary module F then

Hom(F,X)≅ Hom $({}^{F}/_{T}, X)$

But if F is flat then F ≅ $\overset{limit}{\longrightarrow} F_j$ Where F_j is finitely generated and free say of rank n_j

Therefore

Hom(F,X)≅ $Hom(\overset{limit}{\longrightarrow} F_j, X)$

$\cong \overset{limit}{\longleftarrow}$ Hom(F_j,X)

$\cong \overset{limit}{\longleftarrow} Hom(R^{n_j}, X)$

$\cong \overset{limit}{\longleftarrow} (Hom(R, X))^{n_j}$

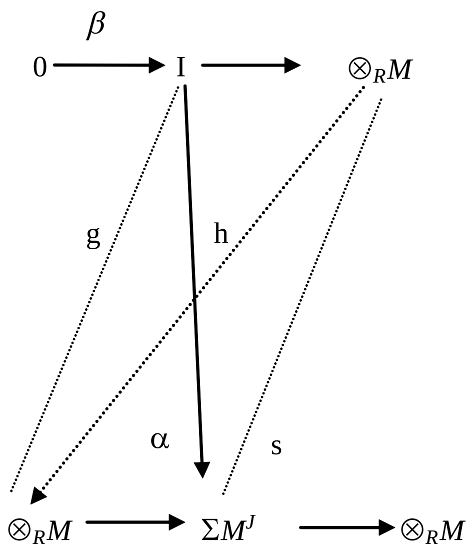
$\cong \overset{limit}{\longleftarrow} X^{n_j}.$

Theorem 3.1: Let M be Strongly M-flat modules then following statements are equivalent

I. Any finitely generated ideal I is almost M –flat modules.
II. Every submodule of M is almost strongly M-flat modules.
III. Every flat module is almost strongly M-flat module.

Proof: This concept we take[24].

I. ⇒ II. Considere the following commutative diagram with I M flat finitely generated ideal.



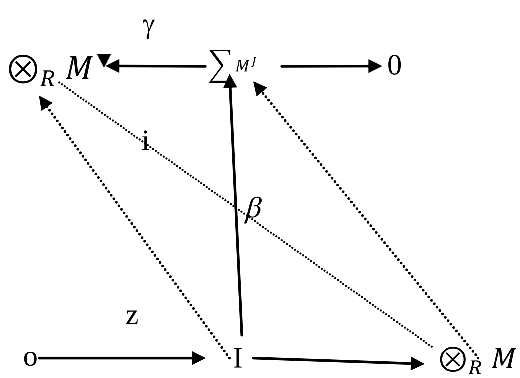
Now I is M-flat ,hence there exists a homomor phism g:I $\longrightarrow \underset{R}{\otimes}$ M such that iog=h

But M is submodule of M then implies the existence of a homomorphism γ:$R\rightarrow M$ *s.t.* $\gamma o \beta$ = g

Hence h=iog=io($\gamma o\beta$) =(i oγ)o β=αoβ

Where α= (i o γ):R$\rightarrow$c hence M^J is strongly M-flat.

II.$\Rightarrow$I. Consider the following commutative diagram with the exact row and I finitely generated



But M^J is almost strongly M-flat hence there exists a homomorphism α:$R\rightarrow M^J$ such that β=αog since R is almost flat then emplies that there exists a homomorphism i:R$\rightarrow$M s.t. α=γ o i Now α=αog=(γ o i)og= γ o (i o g) = γ o h .

Where hog: i$\rightarrow$ M.hence I is almost M flat module.

III$\Rightarrow$ *I* Let $m_i \in$ M and $M^J \in \Sigma$ R such that M=Σ $M^J.m_i$ where i $\leq M^J \leq$ n.

By the dual basis lemma, there exists $\{m_i{:}i\in I\}\subseteq$ Hom(M^J,R) such that for all almost i. and since I is almost M flat module.

III$\Rightarrow$ *II* obviously.

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