Ultra Scientist Vol. 27(1)A, 13-18 (2015).

# On common fixed point theorem in complete metric space for mappings of compatible type $\gamma$

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**(Acceptance Date 3rd February, 2015)**

## Abstract

In this paper we prove a unique common fixed point theorem for pair of self-mappings of weakly compatible maps in a complete metric space for mappings satisfying compatibility of type $\gamma$.

*Key words:* Common fixed point, complete metric space, Compatible maps and weakly compatible maps.

**Subject classification:** 47H10, 54H25.

## 1. Introduction

The concept of the commutatively has generalized in several ways. For this Sessa S [5] has introduced the concept of weakly commuting and Gerald Jungck[3] initiated the concept of compatibility. It can be easily verified that when the two mappings are commuting then they are compatible but not conversely. In 1998, Jungck and Rhoades[4] introduced the notion of weakly compatible and showed that compatible maps are weakly compatible but not conversely. Brian Fisher[1] proved an important Common Fixed Point theorem.

The study of common fixed point of mappings satisfying contractive type conditions has been a very active field of research activity during the last three decades. In 1922, the Polish Mathematician, Banach, proved a theorem which ensures, under appropriate conditions, the existence and uniqueness of fixed point. His result is called Banach fixed point theorem or the Banach contraction principle. This theorem provides a technique for solving a variety of applied problems in mathematical science and engineering. Many authors have extended, generalized and improved Banach fixed point theorem in different ways. Jungck[2] introduced more generalized commuting mappings, called compatible mappings, which are more general than commuting and weakly commuting mappings.

The main purpose of this paper is to present fixed point results for two, four and six maps which has been modified using the concept of weak compatibility and pairs of self maps satisfying new condition compatible type $\gamma$ in complete metric space.

*2. Preliminaries :*

We recall the definitions of complete metric space, the notion of convergence and other results that will be needed in the sequel.

*Definition 2.1:* A metric d on a set X is a function $d: X \times X \to R$ such that for all $x, y \in X$ such that

(i) $d(x, y) \geq 0$ and $d(x, y)=0$ if and only if $x=y$;

(ii) $d(x, y) = d(y, x)$ (symmetry) ;

(iii) $d(x, y) \leq d(x, z)+d(z, x)$ (triangle inequality);

A metric space (X,d) is a set X with a metric d defined on X.

*Definition 2.2:* Let A and B be two self maps on a set X. Maps A and B are said to be commuting if $ABx=BAx$ for all $x \in X$.

*Definition 2.3:* Let A and B be two self maps on a set X. If $Ax=Bx$ for some x in X then x is called coincidence point of A and B.

*Definition 2.4:* Let A and B be two self -maps defined on a set, then A and B are said to be weakly compatible if they commute at coincidence points. That is, $Au=Bu$ for some $u \in X$, then $ABu=BAu$.

*Definition 2.5:* Let A and B be weakly compatible self mappings of a set X. If A and B have a unique point of coincidence, that is $w=Ax=Bx$, that is w is the unique common fixed point of A and B.

*Definition 2.6:* A sequence $\{x_n\}$ in a metric space (X,d) is said to be convergent to a point $x$ if $x \in X$ and $\lim_{n\to\infty} d(x_n, x) = 0$.

*Definition 2.7:* A sequence $\{x_n\}$ in a metric space (X,d) is said to be Cauchy sequence if for every $\varepsilon > 0$ there is $N = N(\varepsilon)$ such that $d(x_m, x_n) < \varepsilon$ for every $m,n>N$.

*Definition 2.8:* A metric space (X,d) is said to be complete if every Cauchy sequence in X is convergent.

*Definition 2.9:* Self mapping A and B of a metric space (X,d) is said to be compatible of type ($\alpha$) if $\lim_{n\to\infty} d(AAx_n, BBx_n)=1$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $x \in X$.

*Definition 2.10:* Self- mappings A and B of a metric space (X,d) is said to be compatible of type ($\gamma$) if $\lim_{n\to\infty} d(ABx_n, BBx_n)=1$ and $\lim_{n\to\infty} d(AA^i x_n, BB^i x_n)=1$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$.

*Lemma 2.11:* Let $(X,d)$ be a metric space. If there exist $k \in (0,1)$ such that $d(x,y) \geq d(x,y)$ for all $x,y$

**3. Main result :**

Let $(X, d)$ be a complete metric space. Suppose that the mappings A, B, C, S, T, R, P and Q are self mappings of X satisfying the following conditions;

(1) $P(X)\subset STR(X)$ and $Q(X)\subset ABC(X)$,

(2) Pairs (P,C), (A,C), (P,B)… are commutative,

(3) Either A,B,C or P is continuous,

(4) The pair [P,ABC] is compatible of type[γ] and [Q,STR] is weak compatible,

(5) $d(Px, Qy) \leq max\{d(Px, ABCx), d(ABCx, STRy), d(STRy, Qy)\}$

Then A, B, C, S, T, R, P and Q have a unique common fixed point in X.

*Proof:*

Let $x_0 \in X$ from condition (1) there exist $x_1$, $x_2 \in X$ such that

$Px_0=STRx_1=y_1$ and $Qx_1=ABCx_2=y_2$

We can find a sequence $\{x_n\}$ and $\{y_n\}$ in X auch that

$Px_{2n}=STRx_{2n+1}=y_{2n}$ and $Qx_{2n+1}=ABCx_{2n+2}=y_{2n+1}$ for n=0,1,2…

If we take $x = x_{2n+1}$ and $y=x_{2n+2}$ all t>0 with q $\in$ (0,1) in condition (5) then sequence $\{y_n\}$ is a Cauchy sequence in X.

Since (X,d) be a complete metric space. Therefore $\{y_n\}\to Z\in X$. Also its subsequence

$$\{ABCx_{2n}\}\to Z \text{ and } \{Px_{2n}\} \to Z \qquad (1)$$

$$\{STRx_{2n+1}\}\to Z \text{ and } \{Qx_{2n}\} \to Z \qquad (2)$$

*Case-I:* BY condition (3) we have ABC is continuous.

Thus $ABC(ABC)^i x_{2n}\to ABCz$ and $ABC(P)x_{2n}\to ABCz$

The pair [P,ABC] is compatible of type (γ),

Therefore

$\lim_{n\to\infty} d\left(PP^i x_{2n}, (ABC)(ABC)^i x_{2n}\right) = 1$ and

$\lim_{n\to\infty} d\left(PP^i x_{2n}, ABCz\right) = 1$

Thus $PP^i x_{2n} \to Z$

Put $x = p^i x_{2n}, y = y_{2n}$ in condition (5) we have

$$d\left(PP^i x_{2n}, Qx_{2n+1}\right) \leq max\left\{d\left(PP^i x_{2n}, (ABC)P^i x_{2n}\right), d\left((ABC)P^i x_{2n}, STRx_{2n+1}\right), d(STRx_{2n+1}, Qx_{2n+1})\right\}$$

Taking $lim_{n\to\infty}$ we have,

$$d(ABCz, z) \leq max\{d(ABCz, ABCz), d(ABCz, z), d(z, z)\}$$

$$\leq max\{1, d(ABCz, z), 1\}$$

$$\leq d(ABCz, z)$$

Therefore from Lemma (2.1)

$d(ABCz,z)=1$

Hence ABCz=z

By putting $x=z$ and $y=x_{2n+1}$ in condition (5) we have,

$$d(Pz, Qx_{2n+1}) \leq max\{d(Pz, ABCz), d(ABCz, STRx_{2n+1}), d(STRx_{2n+1}, Qx_{2n+1})\}$$

Taking $lim_{n\to\infty}$ we have,

$$d(Pz, z) \leq max\{d(Pz, z), d(z, z), d(z, z)\}$$

$$\leq max\{d(Pz, z), 1, 1\}$$

$$\leq d(Pz,z)$$

Such that $d(Pz,z)$

Hence $Pz = z$

Thus

$ABCz = Pz = z$

Put $x$=$Cz$ and $y$=$x_{2n+1}$ in condition (5) we have,

$d(PCz, Qx_{2n+1}) \leq$
$max\{d(PCz, ABC(Cz)), d(ABC(Cz), STRx_{2n+1}),$
$d(STRx_{2n+1}, Qx_{2n+1})\}$

As $PC = CP$, $AC = CA$ and taking $lim_{n\to\infty}$ we have,

$d(Cz, z) \leq max\{d(Cz, Cz), d(Cz, z), d(z, z)\}$
$\leq max\{1, d(Cz, z), 1\}$
$\leq d(Cz, z)$

Such that d(Cz,z)=1

Hence Cz=z

Therefore

$ABCz = Pz = z$
$\Rightarrow ABz = Pz = z$

Putting $x = Bz$ and $y = x_{2n+1}$ in condition (5) we have,

$d(PBz, Qx_{2n+1}) \leq$
$max\{d(PBz, ABC(Cz)), d(ABC(Bz), STRx_{2n+1}),$
$d(STRx_{2n+1}, Qx_{2n+1}))\}$

As $PB = BP$, $AB = BA$ and taking $lim_{n\to\infty}$ we have

$d(Bz, z) \leq max\{d(Bz, Bz), d(Bz, z), d(z, z)\}$
$\leq max\{1, d(Bz, z), 1\}$
$\leq d(Bz, z)$

Such that $d(Bz, z) = 1$

Hence $Bz = z$

Hence

$Az = Pz = z$

Hence $Az = Bz = Cz = Pz = z$

As $P(x) \subset STR(x)$ ther exist $v \in X$ such that $Pz = STRz$

As ther exist such that

Therefore $z = Pz = STRv$

Put $x = x_{2n}$, $y = v$ in condition (5) we have,

$d(Px_{2n}, Qv) \leq max\{d(Px_{2n}, ABCx_{2n}), d(ABCx_{2n},$
$STRv), d(STRv, Qv)\}$

Taking $lim_{n\to\infty}$ we have,

$d(z, Qv) \leq max\{d(z, z), d(z, z), d(z, Qv)\}$
$\leq max\{1,1, d(z, Qv\}\backslash$
$\leq d(z, Qv)$

$\Rightarrow Qv = z$

Therefore $STRv = Qv = z$

As (Q,STR) is weak-compatible we have,

$STRz = z$

Similarly if we put $x = x_{2n}, y = z$ in condition (5) we have,

$d(Px_{2n}, Qz) \leq max\{d(Px_{2n}, ABCx_{2n}), d(ABCx_{2n},$
$STRz), d(STRz, Qz)\}$

Taking $lim_{n\to\infty}$ we have,

$d(z, Qz) \leq max\{d(z, z), d(z, z), d(z, Qz)\}$
$\leq max\{1,1, d(z, Qz)\}$
$\leq d(z, Qz)$

Such that $Qz = z$

Put $x = x_{2n}$, $y = Rz$ in condition (5) we have,

$d(Px_{2n}, Q(Rz)) \leq$
$max\{d(Px_{2n}, ABCx_{2n}), d(ABCx_{2n}, STR(Rz)),$
$d(STR(Rz), Q(Rz))\}$

As $QR = RQ$, $SR = RS$ and taking $lim_{n\to\infty}$ we have,

$d(z, Rz) \leq max\{d(z, z), d(z, Rz), d(Rz, Rz)\}$
$\leq max\{1, d(z, Rz), 1\}$
$\leq d(z, Rz)$

Such that $Rz = z$

Put $x = x_2n$, $y = Tz$ in condition (5) we have,

$d(Px_{2n}, Q(Tz)) \leq max\{d(Px_{2n}, ABCx_{2n}),$

$d(ABCx_{2n}, STR(Rz)), d(STR(Rz), Q(Tz))\}$

As $QT = TQ$, $ST = TS$ and taking $lim_{n\to\infty}$ we have,

$d(z, Tz) \leq d(z, Tz)$

such that $Tz = z$

Therefore $STRz = z\ Qz$

Thus $STRz = STz\ Sz = Qz = z$

We get $Sz = Tz = Rz = z$

Therefore $Az=Bz=Cz=Pz=Sz=Tz=Rz=Qz=z$

Thus z is the common fixed point of *A,B,C,S, T,R,P and Q*

*Case-II:* As P is continuous

$P(ABC)^i x_{2n} \to Pz$ and $PP^i x_{2n} \to Pz$

Since [P,ABC] is compatible of type (γ)

$\lim_{n\to\infty} d\left(PP^i x_{2n}, (ABC)(ABC)^i x_{2n}\right) = 1$ for all t>0

and $\lim_{n\to\infty} d\left(Pz, ABC(ABC)^i x_{2n}\right) = 1$

Therefore $ABC(ABC)^i x_{2n} \to Pz$

Put $x = (ABC)^i x_{2n}, y = x_{2n+1}$ in condition (5) we have,

$d\left(P(ABC)^i x_{2n}, Qx_{2n+1}\right)$

$\leq max\left\{d\left(P(ABC)^i x_{2n}, ABC(ABC)^i x_{2n}\right),\right.$

$\left.d\left(ABC(ABC)^i x_{2n}, STRx_{2n+1}\right), d(STRx_{2n+1}, Qx_{2n+1})\right\}$

Taking $lim_{n\to\infty}$ we have,

$d(Pz, z) \leq d(Pz, z)$

Such that $d(Pz, z) = 1$

Thus $Pz, z$

Hence $Qz=STRz=Sz=Tz=Rz=z$

Since $Q(X)\subset ABC(X)$ there exist $w\in X$ such that $z=Qz=ABCw$

Put $x=w,y=x_{2n+1}$ in condition (5) we have,

$d(Pw, Qx_{2n+1}) \leq max\{d(Pw, ABCw), d(ABCw,$

$STRx_{2n+1}), d(STRx_{2n+1}, Qx_{2n+1})\}$

Taking $lim_{n\to\infty}$ we have,

$d(Pw, z) \leq d(Pw, z)$

Hence $d(Pw, z) = 1$ such that $Pw = z$

Therefore $ABCz=Pz=z$ such that $Bz=z$, $Cz=z$ implies that

$$Az=Bz=Cz=Pz=z$$

Thus z is the common fixed point of A, B, C, S, T, R, P and Q.

*Uniqueness:* Let u be another common fixed point of A, B, C, S, T, R, P and Q.

Then Au=Bu=Cu=Pu=Qu=Su=Tu=Ru=u

Put x=z, y=u in condition (5) we have,

$d(Pz, Qu) \leq max\{d(Pz, ABCz), d(ABCz,$

$STRu), d(STRu, Qu\}$

Which gives z=w. thus z is a unique common fixed point of A,B,C,S,T,R,P and Q.

*Example:* Let x=[0,1] with the metric d defined $d(x,y)=|x-y|$, this is also called induced fuzzy metric space. Induced fuzzy metric space is complete if and only if metric space (X,d) is complete.

$Ax = \begin{Bmatrix} 1\ if\ x = 1 \\ 0\ if\ x \neq 1 \end{Bmatrix}$ $Sx = \begin{Bmatrix} 1\ if\ x\ is\ rational \\ 0\ if\ x\ is\ irrational \end{Bmatrix}$ $Bx=1$ for all $x \in X$

$Tx = \begin{Bmatrix} 1\ if\ x = 1/_2 \\ 0\ if\ x \neq 1/_2 \end{Bmatrix}$ $Ix = \dfrac{x(x+1)}{2}$ $Jx = x$ for all $x \in X$

Since $d(ABx, STy) \geq r(d(Ix, Jy))$ for all $x, y \in X$. Pairs $(AB, I)$ *and* $(ST, J)$ are weakly compatible on X. thus all the conditions of above theorem are satisfied and 1 is common fixed point of A,B,S,T,I and J.

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