Ultra Scientist Vol. 27(1)A, 19-25 (2015).

New Characterizations of T_2 Spaces Using Semi Open Set, Feebly Open Set, Regular Semi Open Set, and Other Related Set Subspaces

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(Acceptance Date 18th January, 2015)

Abstract

Within this paper, T_2 spaces are further characterized using semi open set, feebly open set, regular semi open set, and other related set subspaces.

Key words: T_2 spaces, semi open sets, feebly open sets, subspaces.

AMS Subject Classification: 54A05, 54B05, and 54D10.

1. Introduction

Within this paper, all spaces are topological spaces.

For a topological property questions are naturally raised concerning subspaces of spaces with that property. As a consequence, subspaces and the study of subspaces are included within classical topology. Until recently, the question concerning subspaces for a property P has been "Does a space have property P if and only if every subspace has property P? For subspace properties the proofs of the converse statement are all the same with the property itself only mentioned: "Since the space is a subspace of itself and every subspace has the property, then the space has the property." The feeling that the property itself should have a bigger, more meaningful role in the subspace questions led to the introduction and investigation of proper subspace inherited properties[6].

Definition 1.1. Let (X,T) be a space and let P be a topological property. If (X,T) has property P when every proper subspace of (X,T) has property P, then P is said to be a proper subspace inherited property[6].

Since singleton set spaces satisfy many topological properties, within the recent paper[6] only spaces with three or more elements were considered. Each of the subspace properties T_0, T_1, T_2, Urysohn, regular, and T_3 proved to be proper subspace inherited properties giving new characterizations for each of those properties.

Theorem 1.1. A space (X,T) has property P iff every proper subspace of (X,T) has property P, where P can be each of the properties cited above[6].

The results above raised the question of whether or not topological properties could be further characterized using only certain types of sets within the space as subspaces. With the important role of open and closed sets in the study of topology, a natural place to start such an investigation would be with open set and closed set subspaces, which led to new characterizations of T_0 spaces[7], T_1 spaces[8], T_2 spaces[9], and Urysohn spaces[10].

The successes using open set and closed set subspaces for the T_0 and T_1 separation axioms raised questions about other types of sets in a topological space that could be considered for subspace questions leading to the consideration of regularly open sets.

Regularly open sets were introduced in 1937[23].

Definition 1.2. Let (X,T) be a space and let A be a subset of X. Then A is regularly open, denoted by $A \in RO(X,T)$, if and only if A= Int(Cl(A)).

Within the 1937 paper[23] it was shown that the set of regularly open sets of a space (X,T) forms a base for a topology Ts on X coarser than T and the space (X,Ts) was called the semiregularization space of (X,T). The space (X,T) is semiregular if and only if the set of regularly open sets of (X,T) is a base for T[23].

The introduction of regularly open sets led to the introduction of regularly closed sets.

Definition 1.3. Let (X,T) be a space and let C be a subset of X. Then C is regularly closed, denoted by $C \in RC(X,T)$ iff one of the following equivalent conditions is satisfied: (1) $X \setminus C$ is regularly open and (2) C= Cl(Int(C))[24].

Since their introduction, each of regularly open and regularly closed sets have been further investigated giving additional knowledge and insights in the study of topology. The investigation of subspace questions for regularly open sets[11] led to the following discoveries, which are used in this paper. For a space (X,T), RO(X,T)=RO(X,Ts) and the semiregularization process generates at most one new topology. Thus a space (X,T) is semiregular if and only if T = Ts. For an open set O in a space (X,T), $Cl_T(O)=Cl_{Ts}(O)$ and $(Ts)_O = (T_O)s$, which led to the discovery that semiregular is an open set subspace property, but not a proper open set inherited property. Examples are known showing that semiregular is not a closed set subspace property and that for a closed set C in a space (X,T), $(Ts)_C$ need[11] not be $(T_C)s$.

In a follow-up paper[12], regularly open T_i spaces; i= 0, 1, 2, and regularly open Urysohn spaces were defined by replacing the word open in the definition of T_i; i= 0,1,2, and Urysohn by regularly open, respectively. Within that paper[12], it was shown that a space (X,T) is T_2 iff (X,Ts) is T_2. Within the paper[13], regularly open T_0 spaces were further investigated using subspaces and in the paper[14], regularly open T_1 spaces were further investigated using subspaces. In the paper[9], T_2 spaces were further investigated using convergence and open set, closed set, regularly open set, and regularly closed subspaces. Results in that paper proved to be extremely useful in this paper.

In this paper, T_2 spaces are further characterized using semi open set, feebly open set, semi regular open set, and related set subspaces.

Semi open sets were introduced in 1963 [20].

Definition 1.4. Let (X,T) be a space and let $A \subseteq X$. Then A is semi open, denoted by $A \in$

SO(X,T) iff there exists an $O \in T$ such that $O \subseteq A \subseteq Cl(O)$.

In 1965 semi open sets were further investigated as β-sets and α-sets were introduced[22].

Definition 1.5. Let (X,T) be a space and let $A \subseteq X$. Then A is an α-set, denoted by $A \in \alpha(X,T)$, iff $A \subseteq Int(Cl(Int(A)))$.

In 1970, semi open sets were used to define semi closed sets and the semi closure of a set[1].

Definition 1.6. Let (X,T) be a space and let $A,B \subseteq X$. Then A is semi closed, denoted by $A \in SC(X,T)$, iff X\A is semi open and the semi closure of B, denoted by scl(B), is the intersection of all semi closed sets containing B.

In 1971, the semi closure operator was used to show that for a space (X,T), there is a finest topology F(T) on X which has the same semi open sets as (X,T)[3].

In 1978 the semi closure operator was used to define feebly open sets, which was used to define feebly closed sets and the feebly closure operator[21].

Definition 1.7. Let (X,T) be a space and let $A,B,C \subseteq X$. Then A is feebly open, denoted by $A \in FO(X,T)$, iff there exists an open set O such that $O \subseteq A \subseteq scl(O)$, B is feebly closed, denoted by $B \in FC(X,T)$, iff $X \backslash B \in FO(X,T)$, and the feebly closure of C, denoted by fcl(C), is the intersection of all feebly closed sets containing C.

The further investigation of feebly open sets in 1985 [15] showed for a space (X,T), FO(X,T) is a topology on X and $T \subseteq FO(X,T)=$ FO(X,FO(X,T)). Later in 1985 [16], it was proven that for a space (X,T), $FO(X,T)=\alpha(X,T)=F(T)$.

In 1978 [21], regular semi open sets were introduced.

Definition 1.8. Let (X,T) be a space and let $A \subseteq X$. Then A is regular semi open, denoted by $A \in RSO(X,T)$, iff there exists a $U \in RO(X,T)$ such that $U \subseteq A \subseteq Cl(U)$.

In 1984 [4], the semi interior operator was used, along with the semi closure operator to define semi regular open sets.

Definition 1.9. Let (X,T) be a space and let $A,B \subseteq X$. Then the semi interior of A, denoted by sint(A), is the union of all semi open sets contained in A and B is semi regular open, denoted by $B \in SRO(X,T)$, iff $B = sint(scl(B))$.

In 1987 [5], semi-regular sets were introduced.

Definition 1.10. Let (X,T) be a space and let $A \subseteq X$. Then A is semi-regular, denoted by $A \in SR(X,T)$, iff A is both semi open and semi closed.

Within the 1987 paper [5], it was shown that for a space (X,T), $RSO(X,T) = SRO(X,T) = SR(X,T)$.

Below, the related sets for a space defined above and subspaces are used to further characterized T_2 spaces. As in the cited investigations above, all spaces in this paper will have three or more elements.

2. New Characterizations of T_2 Spaces Using Semi Open Sets, Feebly Open sets, and Regular Semi Open Sets:

Theorem 2.1. Let (X,T) be a space. Then the following are equivalent: (a) (X,T) is T_2, (b) for each $O \in SO(X,T)$, (O,T_O) is T_2, (c) for each $O \in FO(X,T)$, (O,T_O) is T_2, and (d) for each $O \in RSO(X,T)$, (O,T_O) is T_2.

Proof: Clearly, by the results above, (a) implies (b) and since $FO(X,T) \subseteq SO(X,T)$, then (b) implies (c).

(c) implies (d): Since $X \in FO(X,T)$, then (X,T) is T_2 and every subspace of (X,T) is T_2, which implies (d).

Since $X \in RSO(X,T)$, then (d) implies (a).

Within the paper[18], it was shown that a space (X,T) is T_2 iff (X,FO(X,T)) is T_2, which is combined with results above to obtain the following results. Also, the fact that $RO(X,T) \subseteq FO(X,T)) \cap RSO(X,T)$ is used in the result below.

Corollary 2.1. Let (X,T) be a space. Then the following are equivalent: (a) (X,T) is T_2, (b) for each $A \subseteq X$, $(A, FO(X,T)_A)$ is T_2, (b) for each $O \in SO(X,T)$, $(O, FO(X,T)_O)$ is T_2, (c) for each $O \in RO(X,T)$, $(O, FO(X,T)_O)$ is T_2, (d) for each $O \in FO(X,T)$, $(O, FO(X,T)_O)$ is T_2, (e) for each $O \in FO(X,T)$, $(O, (FO(X,T)s)_O)$ is T_2, (f) for each $O \in FO(X,T)$, $(O, ((FO(X,T)_O)s)$ is T_2, and (g) for each $O \in RSO(X,T)$, $(O, FO(X,T)_O)$ is T_2 .

Combining Corollary 2.1 with the fact that $T \subseteq FO(X,T)$ gives the following result.

Corollary 2.2. Let (X,T) be a space. Then the following are equivalent: (a) (X,T) is T_2, (b) for each $O \in T$, $(O, FO(X,T)_O)$ is T_2, (c) for each $O \in T$, $(O, ((FO(X,T))s)_O)$ is T_2, and (d) for each $O \in T$, $(O, ((FO(X,T))_O)s)$ is T_2.

Within the paper[9], the fact that a space (X,T) is T_2 iff (X,Ts) is T_2 was used to further characterize T_2 spaces using subspaces. Below that fact is combined with the results above and the fact that $RO(X,T) \subseteq Ts \subseteq T \subseteq FO(X,T)$ to give additional characterizations of T_2 spaces using subspaces.

Corollary 2.3. Let (X,T) be a space. Then the following are equivalent: (a) (X,T) is T_2, (b) for each $O \in SO(X,Ts)$, $(O, (Ts)_O)$ is T_2, (c) for each $O \in FO(X,Ts)$, $(O, (Ts)_O)$ is T_2, (d) for each $O \in RSO(X,Ts)$, $(O,(Ts)_O)$ is T_2, (e) for each $A \subseteq X$, $(A,(FO(X,T)s)_A$ is T_2, (f) for each $O \in FO(X,T)s$, $(O,(FO(X,T)s)_O)$ is T_2, (g) for each $O \in RO(X,T)$, $(O,(FO(X,T)s)_O)$ is T_2, and (h) for each $O \in SO(X,(FO(X,T)s))$, $(O,((FO(X,T)s)_O)$ is T_2.

<u>*Theorem 2.2.*</u> Let (X,T) be a space. Then the following are equivalent: (a) (X,T) is T_2, (b) for each proper subset $O \in SO(X,T)$, (O,T_O) is T_2 and for each z in X, there exists a proper semi open set containing z, (c) for each proper subset $O \in FO(X,T)$, (O,T_O) is T_2 and for each z in X, there exists a proper feebly open set containing z, and (d) for each proper subset $O \in RSO(X,T)$, (O,T_O) is T_2 and for each z in X, there exists a proper regularly open set containing z.

Proof: By the results above, (a) implies (b).

(b) implies (c): Since each element of FO(X,T) is in SO(X,T), then for each proper subset O in FO(X,T), (O,T_O) is T_2. Let z be in X. Let O be a proper semi open set containing z. Let U be a T-open set such that $U \subseteq O \subseteq Cl(U)$. If $z \in U$, then U is a proper subset in FO(X,T) containing z. Thus consider the case that z is not in U. Let x be in U. Let A and B be disjoint T_O-open sets such that $x \in A$ and $z \in B$. Let $C,D \in T$ such that $A=C \cap O$ and $B=D \cap O$. Then D is a proper T-open set, and thus a proper subset in FO(X,T), containing z.

(c) implies (d): Since $T \subseteq FO(X,T)$, then for each proper subset $O \in T$, (O,T_O) is T_2. Let z be in X. Let O be a proper subset in FO(X,T) containing z. Let U be T-open such that $U \subseteq O \subseteq scl(U)$. If z is in U, then U is a proper T-open set containing z. Thus consider the case that z is not in U. Let $x \in U$. Let A and B be disjoint T_O-open sets such that $x \in A$ and $z \in B$. Let $C,D \in T$ such that $A=O \cap C$ and $B=O \cap D$. Then $C \cap U$ and D are nonempty disjoint T-open sets and Int(Cl(D))is a proper regularly open set containing z. Hence, for each proper T-open set O, (O,T_O) is T_2 and for each z in X, there exists a proper T-open set containing z, which implies (X,T) is T_2 [9]. Thus for each proper subset $O \in RSO(X,T)$, $(O.T_O)$ is T_2.

(d) implies (a): Since $RO(X,T) \subseteq RSO(X,T)$, then for each proper regularly open set O, (O,T_O) is T_2 and for each z in X, there exists a proper regularly open set containing z, which implies (X,T) is T_2.

<u>*Theorem 2.3*</u>. Let (X,T) be a space. Then the following are equivalent: (a) (X,Y) is T_2, (b) for each proper subset $O \in SO(X,T)$, (O,T_O) is T_2 and for each z in X, there exists a proper T-open set containing z, (c) for each proper subset $O \in SO(X,T)$, (O,T_O) is T_2 and for each z in X, there exists a proper feebly open set containing z, (d) for each proper subset $O \in SO(X,T)$, (O,T_O) is T_2 and for each z in X, there exists a proper regularly open set containing z, and (e) for each proper subset $O \in SO(X,T)$, (O,T_O) is T_2 and for each z in X, there exists a proper regular semi open set containing z.

Proof: Clearly (a) implies (b) and, since $T \subseteq FO(X,T)$, (b) implies (c).

(c) implies (d): Since $FO(X,T) \subseteq SO(X,T)$, then for each proper subset $O \in FO(X,T)$,

(O,T_O) is T_2. Since for each z in X, there exists a proper feebly open set containing z, then, by Theorem 2.2, for each z in X, there exists a proper regularly open set containing z.

(d) implies (e): Since $RO(X,T)\subseteq RSO(X,T)$, then for each z in X, there exists a proper regular semi open set containing z.

(e) implies (a): Since $RSO(X,T)\subseteq SO(X,T)$, then for each z in X, there exists a proper semi open set containing z and, by Theorem 2.2, (X,T) is T_2.

Theorem 2.4. Let (X,T) be a space. Then the following are equivalent: (a) (X,T) is T_2, (b) for each proper subset $O\in SO(X,T)$, $(O,(FO(X,T))_O)$ is T_2 and for each z in X, there exists a proper semi open set containing z, (c) for each proper subset $O\in SO(X,T)$, $(O,(FO(X,T))_O)$ is T_2 and for each z in X, there exists a proper feebly open set containing z, (d) for each proper subset $O\in SO(X,T)$, $(O,(FO(X,T))_O)$ is T_2 and for each z in X, there exists a proper regularly open set containing z, and (e) for each proper subset $O\in SO(X,T)$, $(O,(FO(X,T))_O)$ is T_2 and for each z in X, there exists a proper regular semi open set containing z.

Proof: (a) implies (b): Since $SO(X,T)=SO(X,FO(X,T))$ and (X,T) is T_2 iff (X,FO(X,T)) is T_2, then, by Theorem 2.2, (a) implies (b).

Since $FO(X,T)=FO(X,FO(X,T))$, (b) implies (c), since $RO(X,T)=RO(X,FO(X,T))$, then (c) implies (d), and since $RSO(X,T)=RSO(X,FO(X,T))$, (d) implies (e).

(e) implies (a): By the results above and Theorem 2.2, (X,FO(X,T)) is T_2, which implies (X,T) is T_2.

Since a space (X,T) is T_2 iff (X,Ts) is T_2, the following result follows immediately from the results above and Theorem 2.3.

Corollary 2.4. Let (X,T) be a space. Then the following are equivalent: (a) (X,T) is T_2, (b) for each $O\in SO(X,Ts)$, $(O,(Ts)_O)$ is T_2, (c) for each proper subset $O\in SO(X,Ts)$, $(O,(Ts)_O)$ is T_2 and for each z in X, there exists a proper Ts-semi open set containing z, (d) for each proper subset $O\in SO(X,Ts)$, $(O,(Ts)_O)$ is T_2 and for each z in X, there exists a proper Ts- open set containing z, (e) for each proper subset $O\in SO(X,Ts)$, $(O,(Ts)_O)$ is T^2 and for each z in X, there exists a proper Ts-feebly open set containing z, (f) for each proper subset $O\in SO(X,Ts)$, $(O,(Ts)_O)$ is T_2 and for each z in X, there exists a proper Ts-regularly open set containing z, and (g) for each proper subset $O\in SO(X,Ts)$, $(O,(Ts)_O)$ is T_2 and for each z in X, there exists a proper Ts-regular semi open set containing z.

The search for additional subspace characterizations of T_2 spaces led to the consideration of semi open sets of the spaces (X,(FO(X,T))s), (X,FO(X,Ts)), and (X,(FO(X,Ts))s), which led to the results below.

Theorem 2.5. Let (X,T) be a space. Then Ts=(FO(X,T))s=(FO(X,Ts))s) and SO(X,Ts)=SO(X,FO(X,Ts)).

Proof: By Corollary 2.1, Ts=(FO(X,T))s. Replacing Ts by (Ts)s gives (Ts)s= (FO(X,Ts))s. Thus Ts=(Ts)s=(FO(X,Ts))s. As cited earlier, for a space (Y,S), SO(Y,S)=SO(Y,FO(Y,S)). Hence, SO(X,Ts)=SO(X,FO(X,Ts)).

Corollary 2.5. For a space (X,T), SO(X,Ts)=SO(X,(FO(X,T)s)=SO(X,FO(X,Ts))= SO(X,(FO(X,Ts))s).

Thus the consideration of the semi open sets for the cited spaces led to no additional subspace characterizations of T_2 spaces.

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