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Pseudo-Weakly-Finitely-Quasi-injective Modules

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Abstract

In this paper, we generalize the concepts of pseudo weakly-injective (PW-injective for short) modules and weakly-finitely-quasi-injective (WFQ-injective for short) modules to pseudo-weakly-finitely-quasi-injective (PWFQ-injective for short) modules. Our main goal in this work is to study the basic properties of these concepts, characterizations of pseudo-weakly-finitely-quasi-injective modules.

Key words: injective hull E(M), Pseudo-injective, Endomorphism ring, PW-injective modules, WFQ–injective modules, PWFQ-injective modules.

Introduction

Throughout this paper R is an associative ring with identity $1 \neq 0$ and all modules are unitary. Let M and N be right R-modules. A module M is called weakly injective if it is weakly N-injective for each finitely generated module N. A module E is called the injective hull of a module M, if E is an essential extension of M, and E is injective. Here, the base ring is a ring with unity, though possibly non-commutative[9].

Definition 1: A right R-module M is weakly injective if for every finitely generated submodule N of the injective hull E(M) of M there exists $X \subseteq E(M)$ such that $N \subseteq X \cong M$[1].

Lemma 2: Over a right q.f.d. ring R, a module M is weakly injective if and only if each finitely generated module N which is embeddable in E(M) is indeed embeddable in M (*i.e.* M is tight)[1].

A right R-module M is called pseudo-injective if every monomorphism from a submodule of M to M extends to an endomorhism of M [2]. A Right R-module M is called finitely quasi- injective (FQ-injective for short) if every homomorphism from a finitely generated submodule of M to M extends to an endomorphism of M [3]. An R-module M is called pseudo-injective if M is pseudo-M-injective[4]. Let S= End(MR). A ring R is Noetherian if every ideal of R is finitely generated[5].

Definition 3: Let R be a ring. A right R-module M is called pseudo-weakly-finitely-quasi-injective (PWFQ-injective for short) if every monomorphism from a finitely generated submodule of M to M extends to an endomorphism of M. A ring R is called right pseudo-weakly-finitely-injective (right PWF-injective for short) if R_R is PWFQ-injective.

Theorem 4: The following statements are equivalent for a module MR with S=End(MR).

(a) M is PWFQ-injective.

(b) $\mathbf{r}_{R_n}(\alpha) = \mathbf{r}_{R_n}(\beta)$, $\alpha, \beta \in M^n$, $n \in Z^+$ $\Rightarrow S\alpha = S\beta$.

(c) If $a_i \in M$, $i = 1, 2, \ldots\ldots., n$ and $f, \phi : \sum_{i=1}^{n} a_i R \to M$ are monomorphic, then there exists $s \in S$ such that $f = s\phi$.

Proof: (a) $\Rightarrow$ (b).

If $\mathbf{r}_{R_n}(\alpha) = \mathbf{r}_{R_n}(\beta)$, $\alpha, \beta \in M^n$, $n \in Z^+$, where $\alpha=(a_1,a_2,\ldots\ldots\ldots,a_n)$, $\beta=(b_1,b_2,\ldots\ldots\ldots,b_n)$, then the mapping $f : \sum_{i=1}^{n} a_i R \to M$ defined by $f\left(\sum_{i=1}^{n} a_i r_i\right) = \sum_{i=1}^{n} b_i r_i$, where $r_i \in R$, is a monomorphism. Since M is PWFQ-injective, there exists $s \in S$ such that s extends f, then $b_i = f(a_i) = sa_i$, $i = 1, 2, \ldots\ldots., n$, so $\beta = s\alpha$, and thus $S\beta \subseteq S\alpha$. Similarly we can show, $S\alpha \subseteq S\beta$. Hence $S\alpha = S\beta$.

(b) $\Rightarrow$ (c).

Since f, ϕ are monomorphic,

$\mathbf{r}_{R_n}(f(a_1), f(a_2),\ldots\ldots.., f(a_n))$

$= \mathbf{r}_{R_n}(\phi(a_1), \phi(a_2),\ldots\ldots\ldots.,(a_n))$

By (b), we have $S(f(a_1), f(a_2),\ldots\ldots\ldots., f(a_n))$

$= S(\phi(a_1), \phi(a_2),\ldots\ldots\ldots., \phi(a_n))$,

which shows that there exists $s \in S$ such that

$(f(a_1), f(a_2),\ldots\ldots\ldots., f(a_n))$

$= s(\phi(a_1), (a_2),\ldots\ldots\ldots.,\phi(a_n))$,

and hence $f = s\phi$.

(c) $\Rightarrow$ (a).

If $a_i \in M$, $i = 1, 2, \ldots\ldots., n$ and $\phi : \sum_{i=1}^{n} a_i R \to M$ to be the inclusion mapping in (c). So M is PWFQ-injective.

Corollary 5: The following statements are equivalent for a ring R:

(a) R is right PWF-injective.

(b) $\mathbf{r}_{R_n}(\alpha) = \mathbf{r}_{R_n}(\beta)$, $\alpha, \beta \in R^n$, $n \in Z^+ \Rightarrow R\alpha = R\beta$.

(c) If $a_i \in R$, $i = 1, 2, \ldots\ldots., n$ and $f, \phi : \sum_{i=1}^{n} a_i R \to R$ are monomorphic, then there exists $a \in R$ such that $f = a\phi$.

Let M and N be two right R-modules. An R-module M is called finitely N-injective (F-N-injective for short) if for any finitely generated R-submodule A of N and every R-homomorphism from A into M can be extended to an R-homomorphism from N into M, and M is called finitely quasi-injective if M is finitely M-injective[6]. M is said to be pseudo finitely-N-injective (PF-N-injective for short) if for any finitely generated R-submodule H of N, and any R-monomorphism f: H$\to$M can be extended to an R-homomorphism from N into M. An R-module M is called pseudo finitely-injective if M is pseudo finitely-M-injective.

Clearly, M is WFQ-injective if and only if M is WF-M-injective, and M is PWFQ-injective if and only if M is PWF-M-injective.

Remarks 6:
(i). Every pseudo-N-injective module is pseudo-finitely-N-injective for any R-module N.
(ii). Every pseudo-injective R-module is pseudo-weakly-N-quasi-injective module for each R-module N.
(iii). Every pseudo-N-quasi-injective R-module is pseudo-weakly-N-quasi-injective R-module.
(iv). Every weakly-N-injective R-module is pseudo-weakly-N-quasi-injective module.

Proposition 7: Let M and N be two right R-modules then every direct summand of pseudo- weakly-finitely-N-injective (PWF-N-injective for short) R-module is pseudo-weakly- finitely-N-injective.

Proof: Let M be any PWF-N-injective R-module, and A_1 be any direct summand R-submodule of M. then there exists an R-submodule A_1 of M such that $M = A_1 \oplus A_2$. Let B be any finitely generated R-submodule of N, and let $f : B \rightarrow A_1$ be an R- monomorphism (resp., homomorphism). Since M is PWF-N-injective R-module, then f extends to an R-homomorphism $h : N \rightarrow M$. It follows that f extends to a homomorphism of N to A_1 because A_1 is a direct summand of M. Therefore A_1 is PWF-N-injective[7].

Proposition 8: Let M and N be two right R-modules and N_1 be a submodule of N. If M is PWF-N-injective then M is PWF-N_1-injective.

Proof: It is obvious.

Corollary 9: (i). Any direct summand of a PWF-injective R-module is also PWF-injective.
(ii). Any direct summand of a PWFQ-injective module is PWFQ-injective.

Proposition 10: Isomorphic R-module to PWF-N-injective is PWF-N-injective for any R-module N.
Recall that a function f: $N \rightarrow M$ is split, if there exists a function g: $M \rightarrow N$ such that $g\,of = I_N$[9].

Definition 11: Let $\pi : P \rightarrow M$ be an epimorphism. π is called finitely split if for every finitely generated submodule M_o of M there exist a homomorphism $h : M_o \rightarrow P$ such that $\pi(h(x)) = x$, for every $x \in M_o$[10].

Theorem 12: N is a finitely generated R-submodule of an R-module M. If N is PWF-N-injective, then N is a direct summand of M.

Proof: Let N is a finitely generated R-submodule of an R-module M and let $I : N \rightarrow N$ be the identity R-homomorphism. Since N is PWF–N-injective, thus there exists an R- homomorphism $f : M \rightarrow N$ such that $f(a_i) = a_i = I(a_i)$, $\forall\, i = 1, 2, \ldots\ldots\ldots, n$. Then $g : N \rightarrow M$ is the inclusion mapping, hence $(fog)(x) = f(g(x)) = x$, $\forall\, x \in N$. thus g split. Hence N is a direct summand of M [9].

Corollary 13: If N is a finitely generated R-submodule of an R-module M. If N is PWFQ-injective then N is a direct summand of M.

Definition 14: Let M be a right R-module then following[7].

(i). M is called a C1 (or cs or extending) module if every submodule of M is essential in a direct summand of M. It is also equivalent to saying that every closed submodule of M is a direct summand of M.
(ii). M is called a C2 module if every submodule of M that is isomorphic to a direct summand of M is also a direct summand of M.
(iii). M is called a C3 module if direct sum of any two independent direct summands of M is also a direct summand of M.
(iv). M is called a continuous module if M is a C1 and C2 module.
(v). M is called a quasi-continuous module if M is a C1 and C3 module.

We call a module M FC2 if every finitely generated submodule of M that is isomorphic to a direct summand of M is itself a direct summand of M, and we call a module M FC3 if, whenever N and K are direct summands of M with $N \cap K = 0$ and K is finitely generated, then $N \oplus K$ is also a direct summand[8] of M.

Theorem 15: Any PWFQ-injective module is FC2 and FC3.

Proof: Let M be a PWFQ-injective R-module and A be any finitely generated R-submodule of M which is isomorphic to a direct summand B of M. Since M is PWFQ-injective, thus by Prop.7, B is PWF-M-injective, and by prop.10, A is PWF-M-injective. Since A is finitely generated thus A is a direct summand of M by theorem 12. Therefore, M satisfies FC2.

Let M_1 and M_2 be direct summands of M with $M_1 \cap M_2 = 0$ and M_2 is finitely generated. Let S = End(MR). Let $M_1 = eM$ and $M_2 = f M$, where e, f are idempotents in S, then $eM \oplus fM = eM \oplus (1 - e) f M$. Since $(1 - e) f M \cong f M$ is finitely generated, $(1 - e) f M = hM$ for some $h^2 = h \in S$ by FC2. Let $g = e + h - he$, then $g^2 = g$ and $eM \oplus f M = gM$. Thus $M_1 \oplus M_2$ is a direct summand of M, hence M satisfies FC3.

Corollary 16: Let MR be a finitely generated module. Then M is injective if and only if it is weakly injective and PWFQ-injective.

Proof: We need only to prove the sufficiency. Let $a \in E(M)$, then there exists $X \subseteq E(M)$
such that $M + aR \subseteq X \cong M$. Since M is PWFQ-injective, X is also PWFQ-injective. By theorem 15, X is FC2 and hence M is a direct summand of X because M is a finitely generated submodule of X. But $M \subseteq^{ess} E(M)$, so $M \subseteq^{ess} X$. Thus M = X, and then $a \in M$. Therefore, M = E(M), hence M is injective.

Proposition 17: Let M be a finitely generated R-module. Then M is injective if and only if M is PWF-E(M)-injective.

Proof: Let M is injective then clearly M is PWF-E(M)-injective. Conversely suppose M is PWF-E(M)-injective. Let f: $M \to E(M)$ be a monomorphism. Since M is PWF-E(M)-injective, then there exists an R-homomorphism ϕ: $E(M) \to M$ such that $(\phi o f)(x) = \phi(f(x)) = x$, $\forall\ x \in M$ and $\forall\ f(x) \in E(M)$. Therefore, ϕ of = I

Where I: $M \to M$ is the identity R-homomorphism. Therefore f is split. Hence E(M) = f(M) $\oplus A_1$ where A_1 is a R-submodule of E(M). Since E(M) is injective, therefore f(M) is injective[11]. But f(M) $\cong$ M, so M is injective.

Corollary 18: Let M be a finitely generated R-module. Then M is injective if and only if M is PWFQ-injective module.

Proposition 19: If M is a PWF-injective R-module and S=End(M), then SA=SB for each isomorphic R-submodules A,B of M.

Proof: Let f : A$\rightarrow$B be an R-isomorphism. Let b$\in$B then there exists an element a$\in$A such that f (a)=b. It is clear that for each r$\in$R, r a=0 if and only if r b=0. Since M is PWF-injective, then Sb $\subseteq$ Sa, $\forall$ a$\in$A therefore Sb $\subseteq$ SA, $\forall$ b$\in$B. Thus SB $\subseteq$ SA. Similarly, we can show that SA $\subseteq$ SB . Therefore SA=SB.

Corollary 20: If M is a PWFQ-injective R-module and S=End(M), then SA=SB for each isomorphic R-submodules A,B of M.

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