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Gprsg-Homeomorphisms and Sggpr-Homeomorphisms In Topological Spaces

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Abstract

In the present paper we introduce two new types of mappings called gprsg-homeomorphism and sggpr-homeomorphism and then shown that one of these mapping has a group structure. Further we investigate some properties of these two homeomorphisms.

Key words: Homeomorphism; gprsg-homeomorphism; sggpr-homeomorphism.

1. Introduction

The notion homeomorphism plays a very important role in topology. Homeomorphisms between two topological spaces X and Y is a bijective map f: $X \rightarrow Y$ when both f and f^{-1} are continuous. Devi, Balachandran and Maki[11] in 1995 defined two new classes of maps called semi-generalized homeomorphisms and generalized semi-homeomorphisms and also defined two new classes of maps called sgc-homeomorphisms and gsc-homeomorphisms. Ahmed and Narli[15] in 2007 defined two classes of maps called gsg-homeomorphisms and sgs-homeomorphisms. Garg, Chauhan and Agarwal[17] in 2007 introduced two new classes of maps namely gsψ-homeomorphisms and ψgs-homeomorphisms. Garg *et al.*[18] again in 2007 introduced two classes of maps called sgψ-homeomorphisms and ψsg- homeomorphisms. In this paper we introduce two new classes of maps called gprsg-homeomorphisms and sggpr-homeomorphisms and then study some of their properties.

Throughout the present paper, (X, τ) and (Y, σ) denote topological spaces on which no separation axioms are assumed unless explicitly stated. For a subset A of a topological space (X, τ) the cl(A), int(A) and A^C denote the closure of A, the interior of A and the complement of A in X respectively.

2. *Preliminaries :*

In this section we recall the following definitions.

Definition 2.01: A subset A of a topological space (X, τ) is called semi-open[6] (resp. pre-open, regular open) set if A $\subseteq$ cl(int(A)) (resp. A$\subseteq$ int(cl(A)), A= int(cl(A)). The complement of semi open, pre-open, regular open set is called semi-closed, pre-closed,regular-closed respectively.

Definition 2.02: A subset A of a topological space (X, τ) is called semi-generalized closed[7] (briefly sg-closed) if scl(A) $\subseteq$ U whenever A $\subseteq$ U and U is semi-open. The complement of sg-closed set is called sg-open set. Every semi-closed set is sg-closed set. The family of all sg-closed sets of any topological space (X, τ) is denoted by sgc (X, τ).

Definition 2.03: A subset A of a topological space (X, τ) is called generalized semi closed[8] (briefly gs-closed) if scl(A) $\subseteq$ U whenever A$\subseteq$ U and U is open. The complement of gs-closed set is called gs-open set. Every closed (semi-closed, g-closed and sg-closed) set is gs-closed set. The family of all gs-closed sets of any topological space (X, τ) is denoted by gsc (X, τ).

Definition 2.04: A subset A of a topological space (X, τ) is called ψ-closed[12] if scl(A) $\subseteq$ U whenever A $\subseteq$ U and U is sg-open. The complement of ψ-closed set is called ψ-open set. Every closed (semi-closed) set is ψ-closed set and every ψ-closed set is sg-closed (gs-closed) set. The family of all ψ-closed sets of any topological space (X, τ) is denoted by ψc (X, τ).

Definition 2.05: A subset A of a topological space (X, τ) is called generalized pre-regular closed[13] (briefly gpr-closed) set if pcl(A) $\subseteq$ U whenever A$\subseteq$ U and U is reguler-open in (X, τ). The complement of gpr-closed set is called gpr-open set. Every closed set, sg-closed set and gs-closed set is gpr-closed set.The family of all gpr-closed sets of any topological space (X, τ) is denoted by gprc(X, τ).

Definition 2.06: A map f : (X, τ) $\rightarrow$ (Y, σ) is called sg-continuous[4] (resp. gs-continuous[11], ψ-continuous[12], gpr-continuous[0], sg-irresolute[4], gs-irresolute[11], ψ-irresolute[12], gpr-irresolute[( )]) if the inverse image of every closed (resp. closed, closed, closed, sg-closed, gs-closed, ψ-closed, gpr-closed) set in Y is sg-closed (resp. gs-closed, ψ-closed, gpr-closed, sg-closed, gs-closed, ψ-closed, gpr-closed) set in (X, τ).

Definition 2.07: A map f : (X, τ) $\rightarrow$ (Y, σ) is called gsg-irresolute[14] (resp. sgs-irresolute[4], gsψ-irresolute [12], ψgs-irresolute[4], sgψ-irresolute[11], ψsg-irresolute[12]) if the inverse image of every gs-closed (resp. sg-closed, gs-closed, ψ-closed, sg-closed) set in Y is sg-closed (resp. gs-closed, ψ-closed, gs-closed, ψ-closed, sg-closed) set in (X, τ).

Definition 2.08: A bijective map f: (X, τ) $\rightarrow$ (y, σ) is called sgc-homeomorphism[11] (resp. gsc-homeomorphism[11], gsg-homeomorphism[15], sgs-homeomorphism[15], gsψ-homeomorphism[17], ψgs- homeomorphism[17], sgψ-homeomorphism[18], ψsg-homeomorphism[18]) if f and f^{-1} are sg-irresolute (resp. gs-irresolute, gsg-irresolute, sgs-irresolute, gsψ-irresolute, ψgs-irresolute, sgψ-irresolute, ψsg-irresolte).

3. Gprsg-Homeomorphisms :

In this section we introduce gprsg-homeomorphisms and then investigate the group structure of the set of all gprsg-homeomorphisms.

Definition 3.01: A map f: $(X, \tau) \rightarrow (Y, \sigma)$ is called a gprsg-irresolute map if the set $f^{-1}(A)$ is sg-closed in (X, τ) for every gpr-closed set A of (Y, σ).

Definition 3.02: A bijection f: $(X, \tau) \rightarrow (Y, \sigma)$ is called a gprsg- homeomorphisms if the function f and the inverse function f^{-1} are both gprsg -irresolute maps. If there exists a gprsg -homeomorphism from X to Y, then the spaces (X, τ) and (Y, σ) are called gprsg-homeomorphic. The family of all gprsg-homeomorphism of any topological space (X, τ) is denoted by gprsgh-(X, τ).

Remark 3.03: The following examples show that the concepts of homeomorphism and gprsg-homeomorphism are independent of each other.

Example 3.04: Let X = {a, b, c} and $\tau = \{\phi, \{a\}, \{a, b\}, X\}$. Define f: $(X, \tau) \rightarrow (X, \tau)$ by identity mapping then f is a homeomorphism but not a gprsg-homeomorphism.

Example 3.05: Let X = Y = {a, b, c}, $\tau = \{\phi, \{a\}, \{b ,c\}, X\}$ and $\sigma = \{\phi, Y\}$. Define f: $(X, \tau) \rightarrow (Y, \sigma)$ by identity mapping then f is gprsg-homeomorphism but not homeomorphism.

Proposition 3.06: Every gprsg-homeomorphism is (i) sgc-homeomorphism (ii) gsc-homeomorphism (iii) gsg-homeomorphism (iv) sgs-homeomorphism (v) sgψ-homeomorphism (vi) ψgs-homeomorphism (vii) ψsg-homeomorphism.

The converse of the above proposition is not true as at it can be seen from the following examples.

Example 3.07: Let X = Y = {a, b, c}, $\tau = \{\phi, \{a\}, X\}$ and $\sigma = \{\phi, \{a\}, \{a, b\}, Y\}$. Define f: $(X, \tau) \rightarrow (Y, \sigma)$ by identity mapping then f is sgc-homeomorphism and sgs-homeomorphism but not gprsg-homeomorphism.

Example 3.08: Let X = Y = {a, b, c}, $\tau = \{\phi, \{a\}, \{a, b\}, X\}$ and $\sigma = \{\phi, \{a\},\{b\} \{a, b\}, \{a, c\}, Y\}$. Define f: $(X, \tau) \rightarrow (Y, \sigma)$ by identity mapping then f is gsc-homeomorphism but not gprsg-homeomorphism.

Example 3.09: Let X ={a, b, c} and $\tau = \{\phi, \{a\}, \{c\},\{a, c\}, X\}$. Define f: $(X, \tau) \rightarrow (Y, \sigma)$ by identity mapping then f is gsg-homeomorphism and gsψ-homeomorphism but not gprsg-homeomorphism.

Example 3.10: Let X = Y = {a, b, c}, $\tau = \{\phi, \{a\}, \{b, c\}, X\}$ and $\sigma = \{\phi, \{a\},\{c\}, \{a, c\}, Y\}$. Define f: $(X, \tau) \rightarrow (Y, \sigma)$ by identity mapping then f is ψsg-homeomorphism and ψgs-homeomorphism but not gprsg - homeomorphism.

Remark 3.11: gprsg-homeomorphism is independent form sgψ-homeomorphism as it can be seen from the following example.

Example 3.12: Let X = Y = {a, b, c}, $\tau = \{\phi, \{a\},\{a, b\}, \{a, c\}, X\}$ and $\sigma = \{\phi, \{a\}, Y\}$. Define f: $(X, \tau) \rightarrow (Y, \sigma)$ by identity mapping then f is sgψ-homeomorphism but not gprsg-homeomorphism.

Example 3.13: Let X = Y = {a, b, c}, $\tau = \{\phi, \{a\}, \{b, c\}, X\}$ and $\sigma = \{\phi, Y\}$. Define f: $(X, \tau) \to (Y, \sigma)$ by identity mapping then f is gprsg-homeomorphism but not sgψ-homeomorphism.

Theorem 3.14: Every ψgs-homeomorphism, gsψ-homeomorphism, sgs-homeomorphism and gsg-homeomorphism from X onto itself is gprsg-homeomorphism if every gpr-closed set is Ψ-closed set in X.

Theorem 3.15: If f: $(X, \tau) \to (Y, \sigma)$ and g: $(Y, \sigma) \to (Z, \eta)$ are gprsg-homeomorphism then their composition gof: $(X, \tau) \to (Z, \eta)$ is also gprsg-homeomorphism.

Theorem 3.27: If gprsgh(X, τ) is non-empty then the set gprsgh(X, τ) is a group under the composition of maps.

Proof : Define a binary operation $*$: gprsgh$(X, \tau) \to$ gprsgh(X, τ) by $f*g$ = gof for all f, g $\in$ gprsgh(X, τ) and o is the usual operation of compositon of maps then by theorem (3.15) gof $\in$ gprsgh(X, τ). We know that the composition of the maps is associative and the identity element I: $(X, \tau) \to (X, \tau)$ belonging to gprsgh(X, τ) serves as the identity element. If f $\in$ gprsgh(X, τ) then $f^{-1} \in$ gprsgh(X, τ) such that $fof^{-1} = I = f^{-1}of$ and so inverse exists for each element of gprsgh(X, τ). So (gprsgh(X, τ), o) is a group under the operation of composition of maps.

Theorem 3.17: If f: $(X, \tau) \to (Y, \sigma)$ be a gprsg-homeomorphism then f induces an isomorphism from the group gprsgh(X, τ) onto the group gprsgh(Y, σ).

Proof: Define θ_f: gprsgh$(X, \tau) \to$ gprsgh(Y, σ) by $\theta_f(h) = fohof^{-1}$ for every h $\in$ gprsgh((X, τ). Then θ_f is a bijection. Further, for all g_1, $g_2 \in$ gprsgh (X, τ), $\theta_f\,(h_1oh_2) = fo(h_1oh_2)of^{-1} = (foh_1of^{-1})$ o $(fo\,h_2of^{-1}) = \theta_f(h_1)$ o $\theta_f\,(h_2)$. So θ_f is a homeomorphism and so it is an isomorphism induced by f.

4. Sggpr-Homeomorphisms :

In this section we introduce sggpr-homeomorphism and investigate its properties.

Definition 4.01: A map f: $(X, \tau) \to (Y, \sigma)$ is called sggpr-irresolute map is the set $f^{-1}(A)$ is gpr-closed in (X, τ) for every sg-closed set A of (Y, σ).

Definition 4.02: A bijection f : $(X, \tau) \to (Y, \sigma)$ is called a sggpr-homeomorphism if the function f and the inverse function f^{-1} are both sggpr-irresolute maps. If there exists a sggpr-homeomorphism from X to Y, then the spaces (X, τ) and (Y, σ) are called sggpr-homeomorphic.

The family of all sggpr-homeomorphism of any topological space is denoted by sggprh (X, τ).

Theorem 4.03: Every (i) homeomorphism (ii) sgc- homeomorphism (iii) sgs-homeomorphism (iv) gsc-homeomorphism (v) gsg-homeorphism (vi) sgψ-homeomorphism (vii) gsψ-homeomorphism (viii) gprsg-homeomorphism is sggpr- homeomorphism.

The following examples show that the

converse of the above proposition is not true.

Example 4.04: Let X = Y = {a, b, c,}, τ = {ϕ, {b}, {a, b}, {b, c}, X} and σ = {ϕ, Y}. Define f: (X, τ) $\rightarrow$ (Y, σ) by identity mapping then f is sggpr-homeomorphism but not sgc-homeo-morphism, gsc-homeomorphism and sgs-homeomorphism.

Example 4.05 Let X = Y = {a, b, c,}, τ = {ϕ, X} and σ = {ϕ, {a, b}, Y}. Define f: (X, τ) $\rightarrow$ (Y, σ) by identity mapping then f is sggpr-homeomorphism but not homeomorphism and gsg-homeomorphism.

Example 4.06: Let X = Y = {a, b, c,}, τ = {ϕ, {a}, X} and σ = {ϕ, {a, b}, Y}. Define f: (X, τ) $\rightarrow$ (Y, σ) by identity mapping then f is sggpr-homeomorphism but not gsψ-homeomorphism, sgψ-homeomorphism and gprsg-homeomorphism.

Remark 4.07: sggpr-homeomorphism is independent form ψgs-homeomorphism and ψsg-homeomorphism as it can be seen from the following examples.

Example 4.08: Let X = Y = {a, b, c}, τ = {ϕ, {a}, {b, c}, X} and σ = {ϕ, {a}, {a, b}, Y}. Define f: (X, τ) $\rightarrow$ (Y, σ) by identity mapping then f is sggpr-homeomorphism but not ψgs-homeomorphism and ψsg-homeomorphism.

Example 4.09: In example (3.10), map f is ψgs-homeomorphism and ψsg-homeomorphism but not sggpr-homeomorphism.

Theorem 4.10: Every sggpr-homeomorphism from X onto itself is sgc-homeomorphism, sgs-homeomorphism, gsc-homeomorphism, gsg-homeomorphism, sgψ-homeomorphism, gsψ-homeomorphism if every gpr-closed set is ψ-closed set in X.

Theorem 4.11: Every sggpr-homeomorphism from X onto itself is gprsg-homeomorphism if every gpr-closed set is sg-closed set in X.

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