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Symmetry in Plane

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Abstract

In this article we study symmetries of plane figures. We start with the definition of isometry of a metric space. For the better understanding of all the isometries of R^2, we present the known results in form of a flow chart. And then using this idea of isometry, we define symmetry group of a subset of R^2. With the use of this definition we determine symmetry groups of the capital Alphabets of English. We also describe the symmetry group of a regular n-gon which is known as D_n. We also describe subsets of R^2 where symmetry group is the cyclic group of order n which is denoted usually by C_n.

Key words: Symmetry, Isometry.

AMS subject classification: 52B15

Introduction

We begin with definition of isometry: A function f from a metric space (X,d) into a metric space (Y,d') satisfying $d(x,y) = d'(f(x), f(y))$; $\forall\, x, y \in X$ is called an isometry. It is clear that every isometry is continuous and injective. Note that every isometry need not be surjective. However it can be shown that every isometry of R into R is surjective. Note that we have considered R with the usual metric $d(x,y) = |x-y|$. It can be shown that every isometry of R onto R is either a translation or a reflection (reflection in a point).

Composition of two isometries is an isometry. Also we note that the composition of two translations is a translation. Further it can be shown that composition of two reflections from R into R is a translation. Moreover it is interesting to note that every translation is a composition of two reflections. Thus we can say that every isometry of R is a composition of at most two reflections.

We start with understanding isometries of R^2. Note that here we consider the usual

Euclidean metric.

It can be shown that Translations, Rotations, Reflections and Glide reflections (composition of a translation and a reflection in the line parallel to the translation vector) are isometries of R^2. Though not simple, it can be shown that every isometry $f: R^2 \to R^2$ is surjective. In fact every isometry of R^2 is one of the above types (*i.e.* it is either a translation or a rotation or a reflection or a glide reflection). If we accept this fact then surjectivity follows immediately.

If $f: R^2 \to R^2$ is an isometry and $f(\bar{a}) = \bar{a}$, then we say that $\bar{a}$ is a fixed point of *f*. We shall see that fixed points play an important role in classifying isometries. Note that translations don't have any fixed point. Rotations have a unique fixed point, the center of the rotation. Suppose σ^m is a reflection in the line m then each point of the line m is a fixed point of reflection σ^m. Every point is a fixed point of the identity isometry while glide reflection has no fixed point.

Suppose $f: R^2 \to R^2$ is an isometry and $\bar{a} \neq \bar{b} \in R^2$ are fixed points of *f* then every point on the line joining $\bar{a}$ and $\bar{b}$ is also fixed point of *f*. If $f: R^2 \to R^2$ is an isometry having three non-collinear points as fixed points then every point is also a fixed point. And therefore the isometry is identity. Also it is useful to note that isometry of R^2 onto R^2 is completely determined by its values at three non-collinear points.

If $f: R^2 \to R^2$ is an isometry that has two (distinct) fixed points, then it is either a reflection or an identity. If the isometry $f: R^2 \to R^2$ has exactly one fixed point, then it must be a product of two reflections. Every isometry $f: R^2 \to R^2$ is a product of at most three reflections. We summarize these results in the very instructiveFLOWCHART-1 to determine the type of a given isometry of R^2.

We are now in a position to understand plane symmetry in a mathematical way. Let *X*be any subset of R^2. Then, $S(X) = \{f: R^2 \to R^2 / f$ is an isometry and $f(X) = X\}$ is a subgroup of the group of all isometries of R^2 under composition as operation. Here *S(X)* is called the symmetry group of *X* and elements of *S(X)* are called symmetries of *X*.

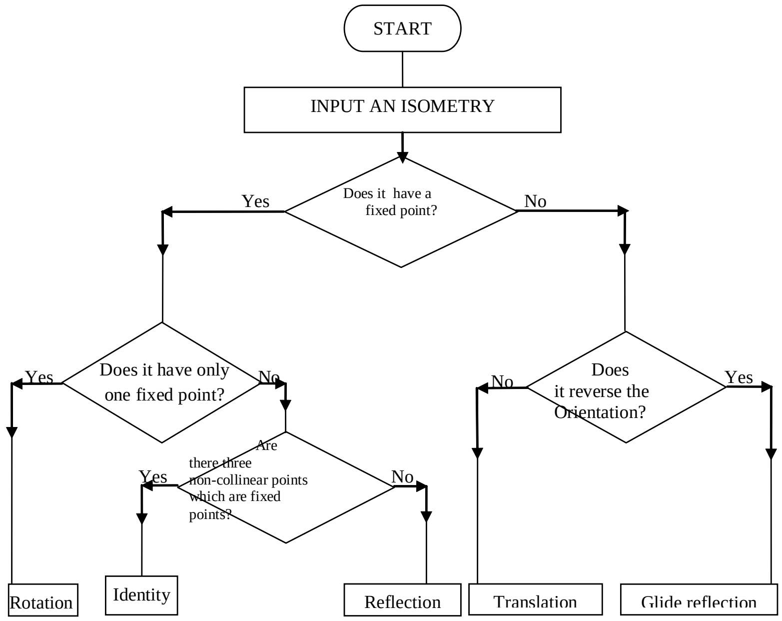
One observes that for more symmetrical objects the group *S(X)* will be large and *S(X)* will be small for less "symmetrical" figures. This can be seen from the fact that if

$$X = S^1 = \{(x, y) \in R^2 / \ x^2 + y^2 = 1\}$$

Is the unit circle, *S(X)* will consist of all rotations about origin and all reflections in all lines passing through origin. Thus *S(X)* is an infinite group when $X = S^1$. If *X* is a scalene triangle then *S(X)* consists of only the identity isometry.

Symmetry Groups of the English alphabets:

We now describe the symmetry groups (see Table 1) of the capital letters of the English alphabets in the form of a table which is very self-explanatory. Note that if you think of 'O' as a perfect circle its symmetry group would be infinite. We think of 'O' as of elliptic shape.



FLOWCHART-1

Table 1.

F, G, J, L, P, Q and R	1) Identity	$\{ I \}$
A, M, T, U, V, W and Y	1) Identity 2) Reflection in the vertical line ℓ chosen appropriately	$\{I, \sigma^{\ell}\}$
B, C, D, E and K	1) Identity 2) Reflection in the horizontal line m chosen appropriately	$\{I, \sigma^{m}\}$
N, S and Z	1) Identity 2) Rotation of angle ? about the point $\circ$ chosen appropriately	$\{I, \rho^{\circ, \pi}\}$
H, I, O and X	1) Identity 2) Reflection in the vertical line ℓ chosen appropriately 3) Reflection in the horizontal line m chosen appropriately 4) Rotation of angle π about the point which is the intersection of line ℓ and m	$\{I, \sigma^{\ell}, \sigma^{m}, \rho^{\circ, \pi}\}$

Table 2.

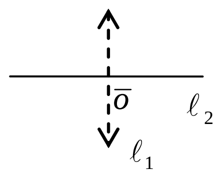
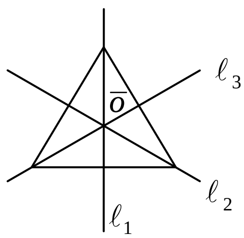
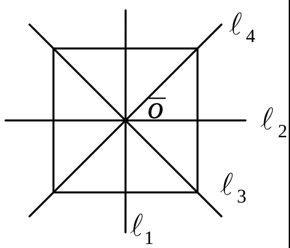
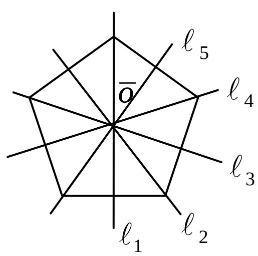
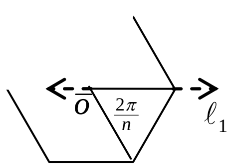
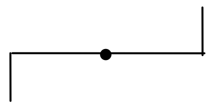
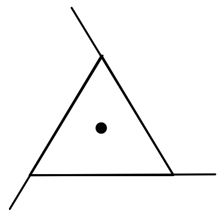
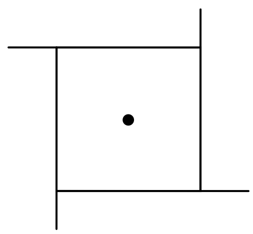
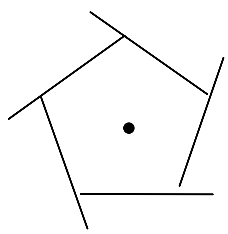
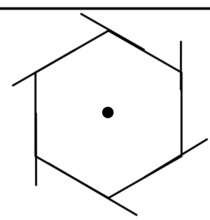
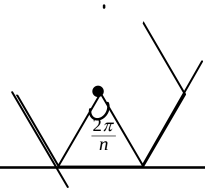
N	Regular n-gon	The symmetry groups S(X)	Order of the group
2		$S(X)=\{I,\ \sigma^{\ell_1},\ \sigma^{\ell_2},\ \rho^{\bar{o},180^\circ}\}$	4
3		$S(X)=\{I,\ \sigma^{\ell_1},\sigma^{\ell_2},\sigma^{\ell_3},\rho^{\bar{o},120^\circ},\ \ldots\ \rho^{\bar{o},240^\circ}\}$	6
4		$S(X)=\{I,\ \sigma^{\ell_1},\sigma^{\ell_2},\sigma^{\ell_3},\sigma^{\ell_4},\rho^{\bar{o},90^\circ},\ \rho^{\bar{o},180^\circ},\ \rho^{\bar{o},270^\circ}\}$	8
5		$S(X)=\{I,\ \sigma^{\ell_1},\sigma^{\ell_2},\sigma^{\ell_3},\sigma^{\ell_4},\sigma^{\ell_5},\ \rho^{\bar{o},72^\circ},\rho^{\bar{o},144^\circ},\ \rho^{\bar{o},216^\circ},\rho^{\bar{o},288^\circ}\}$	10
N		$S(X)=\{I,\ \rho,\ \rho^2,\ldots,\ \rho^{n-1},\ \sigma^{\ell_1},\ \rho\sigma^{\ell_1},\ \rho^2\sigma^{\ell_1},\ldots,\ \rho^{n-1}\sigma^{\ell_1}\}$ Here $\rho^{\bar{o},\frac{2\pi}{n}}=\rho$ and σ^{ℓ_1} is a reflection and $\bar{o}$ is the center of the regular n-gon.	2n

Table 3

Figure X	The symmetry groups S(X)	Order of the group
	$S(X)=\{I,\ \rho^{\bullet,180^\circ}\}$	2
	$S(X)=\{I,\ \rho^{\bullet,120^\circ},\rho^{\bullet,240^\circ}\}$	3
	$S(X)=\{I,\ \rho^{\bullet,90^\circ},\ \rho^{\bullet,180^\circ},\ \rho^{\bullet,270^\circ}\}$	4
	$S(X)=\{I,\ \rho^{\bullet,72^\circ},\ \rho^{\bullet,144^\circ},\rho^{\bullet,216^\circ},\ \rho^{\bullet,288^\circ}\}$	5
	$S(X)=\{I,\ \rho^{\bullet,60^\circ},\ \rho^{\bullet,120^\circ},\ \rho^{\bullet,180^\circ},\ \rho^{\bullet,240^\circ},\ \rho^{\bullet,300^\circ}\}$	6
⋮	⋮	⋮
	$S(X)=\{I,\ \eta,\ \eta^2,\eta^3\ \ldots,\eta^{n-1}\}$ Where $\eta=\rho^{\bullet,\frac{2\pi}{n}}$, so that $\eta^n=I$	n

The group D_n:

Next we present in the form of a table the symmetry groups of regular n-gon. The table is self-contained. However we would like to make the following remarks:

Remark (i): Here the effect of σ^{ℓ_2} and I are same on the 2 – gon (i.e. the line segment). Similarly the effect of σ^{ℓ_1} and $\rho^{\bar{o},\,180^\circ}$ are same on the 2 – gon. However as maps on R^2 they are different in each case.

Remark (ii): Note that the n reflections are written as the product (composition) of rotations ρ^k and the reflection σ^{ℓ_1}.

If n is odd: Then one considers a line through $\bar{o}$ and each vertex. Each of these lines is different if n is odd. We consider reflection in each of these n lines.

If n is even: Here also one first consider a line through $\bar{o}$ and each vertex. But as n is even, each of these lines passes through an opposite vertex. n-gon must have $\frac{n}{2}$ such lines. In addition to these lines, when n is even one can consider lines through the midpoints of a pair of opposite edges. Such lines are also $\frac{n}{2}$ in number. Thus, we have in total n lines, through which we consider reflections.

In any case we have n reflections. And thus *S(X)*, when *X* is a regular n-gon has 2n elements. This group of 2n elements is denoted by D_n.

The cyclic group of order n (C_n) as a symmetry group:

One naturally may ask, are there subsets *X* of R^2 for which $S(X) \cong C_n$ (the cyclic group of order n). The answer is yes. The basic idea is to kill some symmetries which are present in the regular n-gon (and there for in D_n) by adding line segments. We present this information again in a form of a table.

Thus we have seen C_n and D_n as the symmetry groups. Are there any other finite subgroups which may occur as the symmetry groups? The answer is contained in the result due to Leonardo da Vinci[1,2].

Proposition (Leonardo):

If G is a finite subgroup of the group of all isometries of R^2, then G is isomorphic either to C_n or D_n.

References

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2. Shirali, Shailesh, "Symmetry in the world of Man and Nature" Resonance (Vol 6)(6) Page 55 (2001).