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On The Harmonic Summability of Double Fourier Series

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Abstract

In this paper we have proved a theorem on the harmonic summability of double Fourier series, which generalizes various known results. However, our theorem is as follows.

Theorem : If

$$\phi(x,y) = \int_0^x ds \int_0^y \emptyset\,(s,t)dt = o\,(x,y)$$

$$\int_0^\pi dt\,\left|\int_0^\pi (s,t)ds\right| = o\,(x)$$

$$\int_0^\pi dt\,\left|\int_0^\pi \emptyset\,(s,t)ds\right| = o\,(y)$$

m = m (r) > r and n = n (r) > r, such that

$$\int_{y=\frac{1}{n}}^{\frac{1}{(n-r)}} \int_{x=\frac{1}{m}}^{\frac{1}{(m-r)}} \frac{\left|\emptyset\,(x,y) - \emptyset\left(xy - \frac{\pi}{mn}\right)\right|}{xy}\,dxdy$$

$$= o\left(\frac{m}{\left(\log\left(\frac{1}{m}\right)\right)^{\alpha}}\right)\left(\frac{n}{\left(\log\left(\frac{1}{n}\right)\right)^{\alpha}}\right)$$

$$and\ \sum_{\mu=m}^{\infty}\sum_{\nu=n}^{\infty}\left(|a_{\mu\nu}| + |b_{\mu\nu}| + |c_{\mu\nu}| + |d_{\mu\nu}|\right) \to o \quad as\ m,n \to \infty$$

then the double Fourier series of function f(x,y) is summable (H,1,1) to the sum s at x=*u* and y=v.

Key words and phrases : Harmonic summability, Double Fourier series.

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1. Definitions and notations :

Let f(x,y) be a function of f(x,y) periodic with respect to x and with respect to y, in each case with period 2π, and summable in the rectangle Q ($-\pi<x<\pi, -\pi<y<\pi$).

The Fourier series of f(x,y) is given by

$$f(x,y) \sim \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \lambda_{mn}\,[a_{mn} \cos mx \cos ny + b_{mn} \sin mx \cos ny + c_{mn} \cos mx \sin ny + d_{mn} \sin mx \sin ny]$$

where,

$$\lambda_{mn} = \begin{cases} \frac{1}{4}\ for \quad m=0,\ n=0\ \ or\ \ [m=n=0] \\ \frac{1}{2}\ for \quad m>0, n=0\ and\ [m=0, n>0] \\ 1\ for \quad m>0,\ n>0 \end{cases}$$

and

$$a_{mn} = \frac{1}{\pi^2}\int\int_Q f(x,y) \cos mx \cos ny\, dx\, dy$$

with similar expression for $b_{mn}\, c_{mn}$ *and* d_{mn} also

$$\phi(x,y) = \phi(u,v;x,y)$$
$$= \frac{1}{4}\,[\,f(u+x,v+y)+f(u+x,v-y)+f(u-x,v+y)+f(u-x,v-y)-4S\,]$$

$$\phi\,(x,y) = \int_0^x ds \int_0^y |\phi\,(s,t)|\, dt$$

$$P_{[1/x]} = P_x$$

where X = $[1/x]$ is the integral part of 1/x.

$$Q_{[1/y]} = Q_y$$

where Y = [1/y] is the integral part of 1/y

$$\phi\,(x,y) = \int_0^x ds \int_0^y |\phi\,(s,t)|\, dt,$$

$$K_m(x) = \frac{1}{2\pi \log m}\sum_{\nu=0}^{m}\frac{1}{(\nu+1)}\,\frac{\sin\left(m-\nu+\frac{1}{2}\right)x}{\sin\left(\frac{x}{2}\right)}$$

$$K_n(y) = \frac{1}{2\pi \log n}\sum_{\nu=0}^{n}\frac{1}{\nu+1)}\,\frac{\sin\left(n-\nu+\frac{1}{2}\right)y}{\sin\left(\frac{y}{2}\right)}$$

The double sequence $\{s_{mn}\}$ is said to be summable by harmonic mean, or summable (H, 1, 1) if

$$\sigma_{mn} = \lim_{m\to\infty n\to\infty}\left[\frac{1}{(\log m)(\log n)}\sum_{j=0}^{m}\sum_{k=0}^{n}\frac{s_{m-j,n-k}}{(j+1)(k+1)}\right]$$

exists.

1. Introduction

Considering the harmonic summability of Fourier series Hille and Tamarkin[3] have proved the following theorem.

Theorem A.

If

$$\int_0^t |\phi\,(u)|du = o\,(t)$$

$$and\ \lim_{n\to\infty}\frac{1}{\log n}\int_{\frac{\pi}{n}}^{n}\frac{|\phi(t+x)-\phi(t)|}{t}\log\left(\frac{1}{t}\right)dt = 0$$

then the Fourier series is summable (H,1) to the sum zero at the point t=x,

where $x = \frac{\pi}{n}$

Dealing with the topics in this group, Gupta and Gautam[1] have extended the theorem of Hille and Tamarkin in following form

Theorem B.

If

$$\phi\ (t) = \int_0^t |\phi(u)|du = \mathrm{o}\ (t), \quad as \quad t \to 0$$

and further if for any n there is an m=m (n) >n such that

$$\int_{\frac{1}{m}}^{\frac{1}{(m-n)}} \frac{\left|\phi(t) - \phi\left(t - \frac{\pi}{m}\right)\right|}{t} \log\left(\frac{1}{t}\right) dt = \mathrm{o}\ (logn) \qquad as\ n \to \infty$$

$$\sum_{\nu=n}^{\infty} |a_\nu| + |b_\nu| \to \mathrm{o} \qquad as\ n \to \infty$$

then the Fourier series summable (H,1) at the point t=x, where a_0 and b_0 are Fourier coefficients of f.

The main object of this paper is to generalize the above Theorem B, to a more general class of harmonic summability of double Fourier series.

In what follows, we shall prove the following theorem.

1. Theorem :

If

(3.1) $$\phi\ (x,y) = \int_0^x ds \int_0^y \emptyset\ (s,t)dt = \mathrm{o}\ (xy)$$

(3.2) $$\int_0^\pi dt \ \left|\int_0^\pi \emptyset\ (s,t)ds\right| = \mathrm{o}\ (x)$$

(3.3) $$\int_0^\pi ds \ \left|\int_0^\pi \emptyset\ (s,t)ds\right| = \mathrm{o}\ (y)$$

m = m (r) > r and n = n (r) > r, such that

(3.4) $$\int_{y=\frac{1}{n}}^{\frac{1}{(n-r)}} \int_{x=\frac{1}{m}}^{\frac{1}{(m-r)}} \frac{\left|\emptyset\ (x,y) - \emptyset\left(xy - \frac{\pi}{mn}\right)\right|}{xy}\ dxdy$$

$$= \mathrm{o}\left(\frac{m}{\left(\log\left(\frac{1}{m}\right)\right)^{\alpha}}\right)\left(\frac{n}{\left(\log\left(\frac{1}{n}\right)\right)^{\alpha}}\right)$$

and

(3.5) $$\sum_{\mu=m}^{\infty}\sum_{\nu=n}^{\infty}\left(|a_{\mu\nu}| + |b_{\mu\nu}| + |c_{\mu\nu}| + |d_{\mu\nu}|\right) \to 0 \quad as\ m,n \to \infty$$

then the double Fourier series of function f(x,y) is summable (H, 1, 1) to the sum s at x =u and y = v

4. Proof of the theorem :

$$\pi^2\sigma_{mn} = \int_0^\pi \int_0^\pi \emptyset\ (x,y)K_m(x)K_n(y)dx\ dy$$

$$= \left(\int_0^\delta \int_0^\tau + \int_0^\delta \int_\tau^\pi + \int_\delta^\pi \int_0^\tau + \int_\delta^\pi \int_\tau^\pi \right)\emptyset\ (x,y)K_m(x)K_n(y)dxdy$$

$$where \quad \frac{1}{m} < \delta < \pi, \quad \frac{1}{n} < \tau < \pi$$

(4.1) $= I_1 + I_2 + I_3 + I_4$ (say)

Now,

$$I_1 = \int_0^{\delta}\int_0^{\tau} \emptyset\,(x,y)K_m(x)K_n\,(y)dx\,dy$$

$$= \left(\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}} + \int_0^{\frac{1}{m}}\int_{\frac{1}{n}}^{\tau} + \int_{\frac{1}{m}}^{\delta}\int_0^{\frac{1}{n}} + \int_{\frac{1}{m}}^{\delta}\int_{\frac{1}{n}}^{\tau}\right)\emptyset\,(x,y)K_m(x)K_n\,(y)dx\,dy$$

(4.2) $= I_{1,1} + I_{1,2} + I_{1,3} + I_{1,4}$

taking

$$I_{1,1} = \int_0^{I/m}\int_0^{1/n} \emptyset\,(x,y)K_m(x)K_n(y)dx\,dy$$

$$|I_{1,1}| = O\left(\int_0^{1/m}\int_0^{1/n} |\emptyset\,(x,y)|\,|K_m(x)||K_n(y)|dx\,dy\right)$$

$$= O\left(\int_0^{I/m}\int_0^{I/n} |\emptyset\,(x,y)|\left|\frac{I}{2\pi logm}\sum_{\nu=0}^{m}\frac{I}{(\nu+1)}\;\frac{\sin(m-\nu+\frac{1}{2})\,x}{\sin\left(\frac{x}{2}\right)}\right|\right.$$

$$\left.\cdot\left|\frac{I}{2\pi\;logn}\sum_{\nu=O}^{n}\frac{1}{(\nu+1)}\;\frac{\sin\left(n-\nu+\frac{1}{2}\right)y}{\sin\left(\frac{y}{2}\right)}\right|\;dx\,dy\right)$$

$$= O\int_0^{\frac{1}{m}}\int_0^{\frac{1}{m}} |\emptyset\,(x,y)|\left\{O\left(\frac{1}{\left(\log\left(\frac{1}{m}\right)\right)^{\alpha}}\sum_{\nu=o}^{m}\frac{1}{(\nu+1)}\;\frac{(2m-2\nu+1)\sin\left(\frac{x}{2}\right)}{\left|\sin\left(\frac{x}{2}\right)\right|}\right)\right\}$$

$$\left\{O\left(\frac{1}{\left(\log\left(\frac{1}{n}\right)\right)^{\alpha}}\sum_{\nu=0}^{n}\frac{1}{(\nu+1)}\;\frac{(2n-2\nu+1)\sin\left(\frac{y}{2}\right)}{\left|\sin\left(\frac{y}{2}\right)\right|}\right)\right\}\;dx\,dy$$

$$= O\left[\int_0^{I/m}\int_0^{I/n} |\emptyset\ (x,y)| \left\{O\left(\frac{(2m+1)}{\left(\log\left(\frac{I}{n}\right)\right)^{\propto}}\sum_{v=0}^{m}\frac{1}{(v+1)}\right)\right\}\right.$$

$$\left.\left\{O\left(\frac{(2n+1)}{\left(\log\left(\frac{1}{n}\right)\right)^{\alpha}}\sum_{\nu=0}^{n}\frac{1}{(\nu+1)}\right)\right\} dx\,dy\right]$$

$$= O\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}} |\emptyset\ (x,y)|\ \{O\ (m)\}\ \{O\ (n)\}\ dx\ dy$$

$$= O\left[\int_0^{1/m}\int_0^{1/m} |\emptyset\ (x,y)|\ mndxdy\right]$$

(4.3) $\left|\mathrm{I}_{1,1}\right| = \mathrm{o}\ (1)$ by conditions

Now,

$$\mathrm{I}_{1,2} = \int_0^{1/m}\int_{1/n}^{\tau} \emptyset\ (x,y) K_m\ (x) K_n\ (y) dx\ dy$$

$$\left|\mathrm{I}_{1,2}\right| = \left(\int_0^{1/m}\int_{1/n}^{\tau} |\emptyset\ (x,y)|\ |K_m\ (x)|\ \ |K_n\ (y)|\ dx\ dy\right)$$

$$= O\left(\int_0^{1/m}\frac{m}{(\log\left(\frac{1}{m}\right)^{\alpha}}\ dx\ \int_{1/n}^{\tau}\frac{n}{\left(\log\left(\frac{1}{n}\right)\right)^{\alpha}}\ dy\right)$$

(4.4) $\mathrm{I}_{1,2} = \mathrm{o}\ (1)$ by conditions

Again

$$\mathrm{I}_{1,3} = \int_{\frac{1}{m}}^{\delta}\int_{0}^{\frac{1}{n}} \emptyset\ (x,y)K_m\ (x)K_n\ (y)dx\ dy$$

$$\left|\mathrm{I}_{1,3}\right| = \left(\int_{1/m}^{\delta}\int_{0}^{1/n} |\emptyset\ (x,y)|\ |K_m\ (x)|\ |K_n\ (y)|\ dx\ dy\right)$$

$$= 0\left(\int_{1/m}^{\delta} |\emptyset(x,y)|\ |K_m\ (x)|\ \int_{0}^{I/n} \frac{n}{(\log\left(\frac{I}{n}\right)^{\alpha}}\ dy\right)$$

$$= 0\left(\int_{I/m}^{\delta} \frac{m}{(\log\left(\frac{1}{m}\right)^{\alpha}}\ dx\ \int_{0}^{1/m} \frac{n}{\left(\log\left(\frac{1}{n}\right)\right)^{\alpha}}\ dy\right)$$

(4.5) $\left|\mathrm{I}_{1,3}\right| = \mathrm{o}\ (1)$ by conditions

Now,

$$\mathrm{I}_{1,4} = \int_{\frac{1}{m}}^{\delta}\int_{\frac{1}{n}}^{\tau} \emptyset\ (x,y)K_m\ (x)K_n\ (y)dx\ dy$$

$$\left|\mathrm{I}_{1,4}\right| = 0\left(\int_{1/m}^{\delta}\int_{1/m}^{\tau} |\emptyset\ (x,y)|\ \left|\frac{e^{x^2}}{\left(\log\left(\frac{1}{m}\right)\right)^{\alpha}}\ (1+x^2)\right| \left|\frac{e^{y^2}}{\left(\log\left(\frac{1}{n}\right)\right)^{\alpha}}\ (1+y^2)\right|\ dx\ dy\right)$$

$$\left|\mathrm{I}_{1,4}\right| = 0\left(\int_{1/m}^{\delta}\int_{1/n}^{\tau} |\emptyset(x,y)| \left(\frac{e^{x^2}}{\left(\log\left(\frac{1}{m}\right)\right)^{\alpha}}\right)\left(\frac{e^{y^2}}{\left(\log\left(\frac{1}{n}\right)\right)^{\alpha}}\right) dx\ dy\right)$$

$$+ 0\left(\int_{\frac{I}{m}}^{\delta}\int_{\frac{1}{n}}^{\tau} |\emptyset\ (x,y)| \left(\frac{e^{x^2}}{\left(\log\left(\frac{1}{m}\right)\right)^{\alpha}}\right)\left(\frac{xe^{x^2}}{\left(\log\left(\frac{1}{n}\right)\right)^{\alpha}}\right) dx\ \ dy\right)$$

$$+ O\left(\int_{1/m}^{\delta}\int_{1/n}^{\tau} |\emptyset\,(x,y)| \left(\frac{ye^{x^2}}{\left(\log\left(\frac{1}{m}\right)\right)^{\alpha}}\right)\left(\frac{e^{y^2}}{\left(\log\left(\frac{1}{n}\right)\right)^{\alpha}}\right) dx\ \ dy\right)$$

$$+ O\left(\int_{1/m}^{\delta}\int_{1/n}^{\tau} |\emptyset\,(x,y)| \left(\frac{xe^{x^2}}{\left(\log\left(\frac{1}{m}\right)\right)^{\alpha}}\right)\left(\frac{y^{e^{y^2}}}{\left(\log\left(\frac{1}{n}\right)\right)^{\alpha}}\right) dx\ \ dy\right)$$

(4.6) $|\mathrm{I}_{1,4}| = |\mathrm{I}_{1,4,1}| + |\mathrm{I}_{1,4,2}| + |\mathrm{I}_{1,4,3}| + |\mathrm{I}_{1,4,4}|$ (say)

$$|\mathrm{I}_{1,4,1}| = O\left(\int_{1/m}^{\delta}\int_{1/n}^{\tau} |\emptyset\,(x,y)| \left(\frac{e^{x^2}}{\left(\log\left(\frac{1}{m}\right)\right)^{\alpha}}\right)\left(\frac{e^{y^2}}{\left(\log\left(\frac{1}{n}\right)\right)^{\alpha}}\right) dx\, dy\right)$$

$$= O\left(\int_{1/m}^{\delta}\left(\frac{e^{x^2}}{\left(\log\left(\frac{1}{m}\right)\right)^{\alpha}}\right) dx\ \int_{1/n}^{\tau}\left(\frac{|\emptyset(x,y)|e^{y^2}}{\left(\log\left(\frac{1}{n}\right)\right)^{\alpha}}\right) dy\right)$$

$$= O\left[\left(\frac{1}{\left(\log\left(\frac{I}{m}\right)\right)^{\alpha}}\ \int_{1/n}^{\tau} |\emptyset(x,y)|e^{y^2} dy\right)\right]$$

(4.7) $|\mathrm{I}_{1,4,1}| = o(1)$ by conditions

Again

$$|\mathrm{I}_{1,4,2,}| = O\left(\int_{1/m}^{\delta}\int_{1/n}^{\tau} |\emptyset(x,y)| \left(\frac{e^{x^2}}{\left(\log\left(\frac{1}{m}\right)\right)^{\alpha}}\right)\left(\frac{xe^{y^2}}{\left(\log\left(\frac{1}{n}\right)\right)^{\alpha}}\right) dx\, dy\right)$$

$$= O\left(\int_{1/n}^{\tau}\frac{e^{y^2}}{\left(\log\left(\frac{1}{n}\right)\right)^{\alpha}}\, dy\ \int_{1/m}^{\delta} |\emptyset(x,y)| \left(\frac{xe^{x^2}}{\left(\log\left(\frac{1}{m}\right)\right)^{\alpha}}\right) dx\right)$$

$$= O\left(\frac{1}{\left(\log\left(\frac{1}{m}\right)\right)^{\alpha}}\int_{\frac{1}{m}}^{\delta}|\emptyset\,(x,y)|xe^{x^2}dx\right)$$

(4.8) $\left|\mathrm{I}_{1,4,2}\right| = o\,(1)$

$$\left|\mathrm{I}_{1,4,3,}\right| = O\left(\int_{1/m}^{\delta}\int_{1/n}^{\tau}|\emptyset(x,y)|\left(\frac{ye^{x^2}}{\left(\log\left(\frac{1}{m}\right)\right)^{\alpha}}\right)\left(\frac{e^{y^2}}{\left(\log\left(\frac{1}{n}\right)\right)^{\alpha}}\right)dx\,dy\right)$$

$$= O\left(\int_{1/m}^{\delta}\frac{e^{x^2}}{\left(\log\left(\frac{1}{m}\right)\right)^{\alpha}}\,dx\int_{1/n}^{\tau}|\emptyset(x,y)|\left(\frac{ye^{y^2}}{\left(\log\left(\frac{1}{n}\right)\right)^{\alpha}}\right)dy\right)$$

$$= O\left(\frac{1}{\left(\log\left(\frac{1}{n}\right)\right)^{\alpha}}\int_{\frac{1}{n}}^{\tau}|\emptyset\,(x,y)|ye^{y^2}dy\right)$$

(4.9) $\left|\mathrm{I}_{1,4,3}\right| = o\,(1)$

Again

$$\left|\mathrm{I}_{1,4,4}\right| = O\left(\int_{1/m}^{\delta}\int_{1/n}^{\tau}|\emptyset(x,y)|\left(\frac{xe^{x^2}}{\left(\log\left(\frac{1}{m}\right)\right)^{\alpha}}\right)\left(\frac{ye^{y^2}}{\left(\log\left(\frac{1}{n}\right)\right)^{\alpha}}\right)dx\,dy\right)$$

$$= O\left(\int_{1/m}^{\delta}|\emptyset(x,y)|\left(\frac{xe^{x^2}}{\left(\log\left(\frac{1}{m}\right)\right)^{\alpha}}\right)dx\int_{1/n}^{\tau}\left(\frac{ye^{y^2}}{\left(\log\left(\frac{1}{n}\right)\right)^{\alpha}}\right)dy\right)$$

$$= O\left(\frac{1}{\left(\log\left(\frac{1}{n}\right)\right)^{\alpha}}\int_{\frac{1}{m}}^{\delta}|\emptyset\,(x,y)|\left(\frac{xe^{x^2}}{\left(\log\left(\frac{1}{m}\right)\right)^{\alpha}}\right)dx\right)$$

(4.10) $|I_{1,4,4}| = o(1)$ by conditions

Combining (4.6), (4.7), (4.8), (4.9) and (4.10)

(4.11) $|I_{1,4}| = o(1)$

Considering (4.2), (4.3), (4.4), (4.5) and (4.11)

(4.12) $|I_1| = o(1)$

Now,

$$I_2 = \int_0^{\delta}\int_{\tau}^{\pi} \emptyset(x,y)K_m(x)K_n(y)dx\,dy$$

$$= \int_{\tau}^{\pi} K_m(x)dx \int_0^{\delta} \emptyset(x,y)K_n(y)dy$$

$$= \int_{\tau}^{\pi} K_m(x)dx \int_0^{\frac{1}{n}} \emptyset(x,y)K_n(y)dy$$

$$+ \int_{\tau}^{\pi} K_m(x)dx \int_{1/n}^{\delta} \emptyset(x,y)K_n(y)dy$$

(4.13) $I_2 = I_{2,1} + I_{2,2}$ (say)

$$|I_{2,1}| = \left|\int_{\tau}^{\pi} K_m(x)dx \int_0^{1/n} \emptyset(x,y)K_n(y)dy\right|$$

$$= o\left[\left|\frac{\pi}{\left(\log\left(\frac{1}{m}\right)\right)^{\alpha}\left(\log\left(\frac{1}{n}\right)\right)^{\alpha}} \int_0^{1/n} \emptyset(x,y)dy\right|\right]$$

(4.14) $|I_{2,1}| = o(1)$

Now

$$|I_{2,2}| = \left|\int_{\tau}^{\pi} K_m(x)dx \int_{I/n}^{\delta} \emptyset(x,y)K_n(y)dy\right|$$

$$= o\left(\frac{1}{\left(\log\left(\frac{1}{m}\right)\right)^{\alpha}} \int_{\frac{I}{n}}^{\delta} \emptyset(x,y)\frac{(1+y^2)}{y\left(\log\left(\frac{1}{n}\right)\right)^{\alpha}}\,dy\right)$$

$$= o\left(\frac{1}{\left(\log\left(\frac{1}{m}\right)\right)^{\alpha}\left(\log\left(\frac{1}{n}\right)\right)^{\alpha}} \int_{I/n}^{\delta} \frac{|\emptyset(x,y)|}{y}\,dy\right)$$

$$+ o\left(\frac{1}{\left(\log\left(\frac{1}{m}\right)\right)^{\alpha}\left(\log\left(\frac{1}{n}\right)\right)^{\alpha}} \int_{I/n}^{\delta} |\emptyset(x,y)|ye^{y^2}dy\right)$$

(4.15) $|I_{2,2}| = |I_{2,2,1}| + |I_{2,2,2}|$ (say)

Now

$$|I_{2,2,1}| = o\left(\frac{1}{\left(\log\left(\frac{1}{m}\right)\right)^{\alpha}\left(\log\left(\frac{1}{n}\right)\right)^{\alpha}} \int_{I/n}^{\delta} \frac{|\emptyset(x,y)|}{y}\,dy\right)$$

$$= o\left(\frac{1}{\left(\log\left(\frac{1}{m}\right)\right)^{\alpha}\left(\log\left(\frac{1}{n}\right)\right)^{\alpha}} \left\{\emptyset(x,y)e^{y^2}\right\}_{1/n}^{\delta}\right.$$

$$\left. + \frac{1}{\left(\log\left(\frac{1}{m}\right)\right)^{\alpha}\left(\log\left(\frac{1}{n}\right)\right)^{\alpha}} \int_{1/n}^{\delta} e^{2y^2}\emptyset(x,y)dy\right)$$

$$= o\left(\frac{1}{\left(\log\left(\frac{1}{m}\right)\right)^{\alpha}\left(\log\left(\frac{1}{n}\right)\right)^{\alpha}}\right)$$

$$+ o\left(\frac{1}{\left(\log\left(\frac{1}{m}\right)\right)^{\alpha}\left(\log\left(\frac{1}{n}\right)\right)^{\alpha}} \int_{I/n}^{\delta} e^{2y^2}\emptyset(x,y)dy\right)$$

$$= o(1) + o\left(\frac{\log y^2}{n}\right)_{1/n}^{\delta}$$

(4.16) $|I_{2,2,1}| = o(1)$ by conditions

Again

$$|I_{2,2,2}| = O\left(\frac{1}{\left(\log\left(\frac{1}{m}\right)\right)^{\alpha}\left(\log\left(\frac{1}{n}\right)\right)^{\alpha}} \int_{1/n}^{\delta} |\emptyset(x,y)|ye^{y^2}\,dy\right)$$

$$= O\left(\frac{1}{\left(\log\left(\frac{1}{m}\right)\right)^{\alpha}\left(\log\left(\frac{1}{n}\right)\right)^{\alpha}} \left\{\emptyset(x,y)ye^{y^2}\right\}_{1/n}^{\delta}\right.$$

$$\left. + \frac{1}{\left(\log\left(\frac{1}{m}\right)\right)^{\alpha}\left(\log\left(\frac{1}{n}\right)\right)^{\alpha}} \int_{I/n}^{\delta} \emptyset(x,y)(1+y^2)e^{2y^2}dy\right)$$

$$= o\,(1) + O\left(\frac{1}{\left(\log\left(\frac{1}{m}\right)\right)^{\alpha}\left(\log\left(\frac{1}{n}\right)\right)^{\alpha}}\int_{I/n}^{\delta} e^{2y^2}\,\emptyset\,(x,y)dy\right)$$

$$+\,O\left(\frac{1}{\left(\log\left(\frac{1}{m}\right)\right)^{\alpha}\left(\log\left(\frac{1}{n}\right)\right)^{\alpha}}\int_{I/n}^{\delta} ye^{2y^2}\,\emptyset\,(x,y)dy\right)$$

$$= o\,(1) + O\left(\frac{1}{\left(\log\left(\frac{1}{n}\right)\right)^{\alpha}}\int_{1/n}^{\delta}\left(\frac{e^{y^2}}{y}\right)dy\right)$$

$$+\,O\left(\frac{1}{\left(\log\left(\frac{1}{n}\right)\right)^{\alpha}}\int_{1/n}^{\delta} e^{y^2}\,dy\right)$$

$$= o\,(1) + O\left(\left\{\frac{\log y^2}{\left(\left(\log\left(\frac{1}{n}\right)\right)^{\alpha}\right.}\right\}_{1/n}^{\delta}\right) + O\left(\left\{\frac{y}{\left(\left(\log\left(\frac{1}{n}\right)\right)^{\alpha}\right.}\right\}_{1/n}^{\delta}\right)$$

(4.17) $\left|I_{2,2,2}\right| = o\,(1)$

Considering (4.13), (4.14), (4.15) (4.16) and (4.17)

(4.18) $|I_2| = o\,(1)$

Similarly

(4.19) $|I_3| = o\,(1)$

$$|I_4| = \int_{\delta}^{\pi}\int_{\tau}^{\pi}\emptyset\,(x,y)K_m\,(x)K_n\,(y)dx\,dy$$

$$|I_4| = O\left(\frac{1}{\left(\log\left(\frac{1}{m}\right)\right)^{\alpha}\left(\log\left(\frac{1}{n}\right)\right)^{\alpha}}\int_{\delta}^{\pi}\int_{\tau}^{\pi}|\emptyset(x,y)|\,e^{x^2}\,e^{y^2}\,dx\,dy\right)$$

(4.20) $|I_4| = o\,(1)$ by conditions

Finally the collection of (4.1), (4.12), (4.18) (4.19) and (4.20) completes the proof of the theorem.

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