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Vertex-edge Matrix of fuzzy Graphs

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Abstract

In this paper for the first time two new notions in fuzzy graphs viz., vertex-edge matrix (v-e matrix) of a fuzzy graph and dominant fuzzy graph graphs are introduced. Several properties enjoyed by them are enumerated. Some interesting results in this direction is obtained.

Key words: Fuzzy graphs, dominant fuzzy graphs, strict fuzzy graphs, vertex-edge matrix (v-e matrix) of a fuzzy graph.

Introduction

This paper introduces two new concepts related with fuzzy graphs and proves some interesting properties about them. This paper has three sections. Section one is introductory in nature. Section two introduces the new notion regarding fuzzy graphs and derives a few interesting properties about them. The final section gives conclusions based on our study.

1 Preliminaries :

In this section we recall the basic properties[1,2].

Definition 1.1 The *scalar cardinality* of a fuzzy set μ defined on a finite universal set X is denoted by $|\mu|$ and is defined as $|\mu| = \sum_{x\in X}\mu(x)$.

Definition 1.2 A *fuzzy graph* (undirected and without loops) $G = (V, \mu, \rho)$ is a non-empty finite set V together with a pair of functions $\mu: V \to [0,1]$ and $\rho: V \times V \to [0,1]$ such that $\forall\, u, v \in V, \rho(u, v) = \rho(v, u) \leq min\{\mu(u), \mu(v)\}$ and $\rho(u, u) = 0$.

μ is called the *fuzzy vertex set* and ρ is called the *fuzzy edge set*.

Definition 1.3 A fuzzy vertex $u \in V$ is called an *active vertex* if $\mu(\mu)$>0. A fuzzy graph is said to be an *active fuzzy graph* if all its fuzzy vertices are active. That is, the support of the function μ is the whole of V.

Note: Throughout our discussion we consider only active fuzzy graphs without explicitly mentioning it. We also write $G=(V,\mu,\rho)$ as $G=(\mu,\rho)$ when the underlying vertex set is clear from the context.

Definition 1.4 A *path P* in a fuzzy graph $G=(V,\mu,\rho)$ is a sequence of distinct vertices (except possibly u_0 and u_n) $u_0, u_1, u_2, \ldots u_n$ such that $\rho(u_{i-1}, u_i) > 0, 1 \leq i \leq n$. $n \geq 1$ is called the *length of the path*.

We call P a cycle if $u_0 = u_n$ and n ≥ 3.

Definition 1.5 Consider the Cartesian product $G = G_1 \times G_2 = (V, X)$ of graphs $G_1 = (V_1, X_1)$ and $G_1 = (V_2, X_2)$. Then $V = V_1 \times V_2$ and $X = \{(u, u_2)(u, v_2) | u \in V_1, u_2 v_2 \in X_2\} \cup \{(u_1, w)(v_1, w) | w \in V_2, u_1 v_1 \in X_1\}$.

Let μ_i be a fuzzy subset of V_i and , ρ_i a fuzzy set of $X_i, i = 1,2$. Define fuzzy subsets $\mu_1 \times \mu_2$ of V and $\rho_1\rho_2$ of X as follows:

$$\forall (u_1, u_2) \in V, (\mu_1 \times \mu_2)(u_1, u_2) = min\{\mu_1(u_1), \mu_2(u_2)\};$$

$$\forall u \in V_1, \forall u_2 v_2 \in X_2, \rho_1\rho_2\big((u, u_2)(u, v_2)\big) = min\{\mu_1(u), \rho_2(u_2 v_2)\};$$

$$\forall w \in V_2, \forall u_1 v_1 \in X_1, \rho_1\rho_2((u_1, w)(v_1, w)) = min\{\mu_2(w), \rho_1(u_1 v_1)\}.$$

Let (μ_i, ρ_i) be a partial fuzzy subgraph of $G_i, i=1,2$. Then $((\mu_1 \times \mu_2), \rho_1\rho_2)$ is a partial fuzzy subgraph of G and is called the Cartesian product of (μ_1, ρ_1) and (μ_2, ρ_2).

Theorem 1.1 If $A=[a_{ij}]$ is a symmetric $n \times n$ strictly diagonally dominant matrix with positive diagonal entries then A is positive definite.

Definition 1.6 A fuzzy graph $G=(V,\mu,\rho)$ is said to be complete if $\rho(u, v) = min\{\mu(u), \mu(v)\}\ \forall u, v \in V$.

2 Definition of vertex-edge matrix (v-e matrix) of a fuzzy graph and its properties:

In this section for the first time we define the notion of a v-e matrix of a fuzzy graph. We make a list of properties enjoyed by these new v-e matrices. Further we prove v-e matrices obtained by different orderings of the vertices lead only to the same characteristic equation. This is also represented by some examples. Further the new definition of dominant fuzzy vertex is introduced. This dominant fuzzy vertex also satisfies certain list of properties which are enumerated in this section. We prove the v-e matrix of a fuzzy graph with dominant fuzzy vertices is positive definite. Also we prove, if two fuzzy graphs have their fuzzy vertices to be dominant then their Cartesian product need not in general be a dominant fuzzy graph.

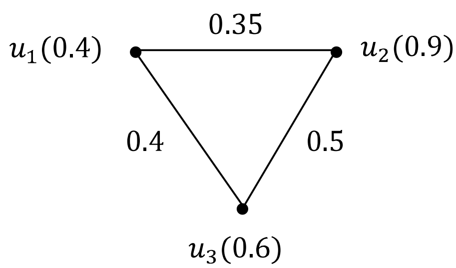
We now proceed to define the vertex-edge matrix of a fuzzy graph.

Definition 2.1 Let $G=(V,\mu,\rho)$ be a fuzzy graph with $|V|=n$. Let $V = \{u_1, u_2, \ldots, u_n\}$ be an ordered set of vertices. The *vertex-edge matrix of the fuzzy graph* G with respect to the ordered set V is denoted by $m_V(G) = [a_{ij}]_{n \times n}$ *where*

$$a_{ij} = \begin{cases} \rho(u_i, u_j) \; if \; i \neq j \\ \mu(u_i) \; if \; i = j \end{cases}.$$

Thus, in this way, any given fuzzy graph may be represented by a matrix. Also, given any symmetric v-e matrix with entries from [0,1] such that the diagonal entries are greater than or equal to the corresponding entries in the rows (and hence columns because of symmetry), we can have the associated fuzzy graph. All these are represented by the following examples.

Example 2.1 Consider the following fuzzy graph *G*



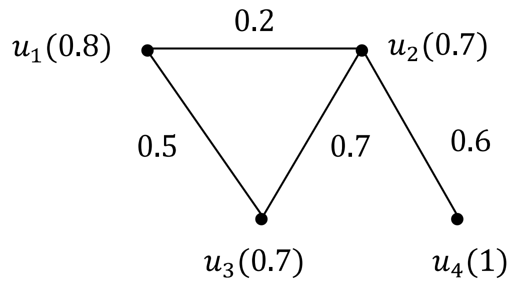
The v-e matrix $m_V(G)$ associated with this graph *G* is as follows:

$$m_V(G) = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \end{array} \begin{array}{c} \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} \\ \begin{bmatrix} 0.4 & 0.35 & 0.4 \\ 0.35 & 0.9 & 0.5 \\ 0.4 & 0.5 & 0.6 \end{bmatrix} \end{array}.$$

Example 2.2 Consider the following symmetric matrix *M* with entries from [0,1] such that each diagonal entry is positive and is greater than or equal to the corresponding entries in each row(and hence column);

$$M = \begin{bmatrix} 0.8 & 0.2 & 0.5 & 0 \\ 0.2 & 0.7 & 0.7 & 0.6 \\ 0.5 & 0.7 & 0.7 & 0 \\ 0 & 0.6 & 0 & 1 \end{bmatrix}.$$

Its corresponding fuzzy graph is given in the following.



The following results are true.

Result 2.1 A v-e matrix of a fuzzy graph is symmetric since $\rho(u_i, u_j) = \rho(u_j, u_i) \; \forall \; 1 \leq i, j \leq n$.

Result 2.2 By the definition of fuzzy graph, we have $\rho(u_i, u_j) \leq \min\{\mu(u_i), \mu(u_j)\} \; \forall \; 1 \leq i, j \leq n$ and therefore, the diagonal entries are greater than or equal to any other entries in the respective rows and columns.

Result 2.3 If $G=(V,\mu,\rho)$ is a path and *V* is an ordered set of vertices such that they are ordered in the order of the path (sequence of vertices in the path), then the matrix of *G* with respect to this ordered vertex set is a tridiagonal v-e matrix.

Result 2.4 The trace of a v-e matrix of *G* gives the scalar cardinality of its fuzzy vertex set.

Result 2.5 Since the v-e matrix of a fuzzy graph is a real symmetric matrix, all its eigen values are real.

Result 2.6 Every v-e matrix of a fuzzy graph is diagonalizable.

We now discuss about the properties of v-e matrices of G corresponding to different ordered vertex sets.

Let $G=(V,\mu,\rho)$ be a fuzzy graph with $|V|=n$. The n vertices of V can be ordered in $n!$ ways. And so there are $n!$ v-e matrices (not necessarily distinct), all of which would represent the same fuzzy graph G.

Proposition 2.1 Let $G=(V,\mu,\rho)$ be a fuzzy graph with $|V|=n\geq 2$ and let A and B denote two v-e matrices of G obtained from two different ordering of the vertices. Then A can be obtained from B (and vice-versa) by a series of alternate row and column interchanges.

Theorem 2.1 Let A and B denote two v-e matrices of $G=(V,\mu,\rho)$ obtained from two different ordering of the vertices. Then A and B have the same characteristic equation.

Proof The characteristic matrix of B is $[B-\lambda I]$ and it can be obtained by a series of alternate row and column interchanges of $[B-\lambda I]$, and therefore its determinant value is unaltered. This is because when two rows (or columns) of a determinant are interchanged its sign is interchanged. However, corresponding to each row interchange, we have a column interchange also hence this double negation makes no change in the determinant value of the characteristic matrix. Thus the determinant values of these matrices are the same. Or equivalently, they have the same characteristic equations.

Corollary 2.1 :
i. $tr\ A = trB$
ii. $\det A = trB$
iii. The eigen values of A and B are the same.

The following results can be easily proved

Theorem 2.2: Let $G=(V,\mu,\rho)$ be a fuzzy graph with $|V|=n\geq 2$ and let A and B denote two v-e matrices of G obtained from two different ordering of the vertices. Then A and B are similar.

Proof : By theorem 2.1, A and B have the same characteristic equation and hence the same eigen values. Let $\lambda_1, \lambda_2, \ldots \lambda_n$ be the eigen values of A (and hence B also). By Result 2.6 both A and B are diagonalizable. Let A be similar to a diagonal matrix D and B be similar to a diagonal matrix say D'. Without loss of generality, assume that $D = diag(\lambda_1\ \lambda_2, \ldots \lambda_n)$. Then $D' = diag\left(\lambda_{\sigma(1)},\ \lambda_{\sigma(2)}, \ldots, \lambda_{\sigma(n)}\right)$ where $\sigma(1), \sigma(2), \ldots, \sigma(n)$ is a permutation of 1, 2, 3, …, n. Let S and T be non-singular matrices such that $D = S^{-1}AS$ and $D' = T^{-1}BT$.

Note that the matrices D and D' are similar. That is, there exists a non-singular matrix U such that $D' = U^{-1}DU$. Now, $B = TD'T^{-1} = T(U^{-1}DU)T^{-1} = TU^{-1}$ $(S^{-1}AS)UT^{-1} = (TU^{-1}S^{-1})A(SUT^{-1})$ $= (TU^{-1}S^{-1})A(TU^{-1}S^{-1}))^{-1}$. Thus B is similar to . Hence the theorem.

Result 2.7 It is not necessary that all the $n!$ v-e matrices are distinct. This is illustrated by the following example.

Example 2.3 Let $A = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$ and $B = \begin{bmatrix} b & h \\ h & a \end{bmatrix}$ be the two v-e matrix of a fuzzy graph with two fuzzy vertices u_1 and u_2. If $a=b$ then $A=B$. So in this case the 2! = 2 matrices are the same.

Here the new concept of strict fuzzy graph is given.

Definition 2.2 A fuzzy graph $G=(V,\mu,\rho)$ is said to be a *strict fuzzy graph* if $\rho(u_i,u_j) < \min\{\mu(u_i),\mu(u_j)\}\ \forall\ 1 \le i,j \le |V|$.

The following results are direct.

Result 2.8 A strict fuzzy graph $G=(V,\mu,\rho)$ with $|V|>1$ is not complete. (For, if G is a strict fuzzy graph then $\rho(u_i,u_j) < \min\{\mu(u_i),\mu(u_j)\}\ \forall\ 1 \le i,j \le |V|$ and so $\rho(u_i,u_j) \ne \min\{\mu(u_i),\mu(u_j)\}\ \forall\ 1 \le i,j \le |V|$.)

Result 2.9 The diagonal entries of a strict fuzzy graph are strictly greater than all other entries in the respective rows and columns.

Next we proceed onto define the notion of dominant fuzzy vertex.

Definition 2.3 A fuzzy vertex u_k in a fuzzy graph $G=(V,\mu,\rho)$ is said to be a *dominant (fuzzy) vertex* if

$$\mu(u_k) > \sum_{1\le j\le |V|} \rho(u_k,u_j).$$

Result 2.10 Every dominant vertex is an active vertex. (For, if u_k is a dominant vertex then $\mu(u_k) > \sum_{1\le j\le |V|} \rho(u_k,u_j) \ge 0$ and hence $\mu(u_k) > 0$).

Result 2.11 A complete fuzzy graph $G=(V,\mu,\rho)$ with $|V|>1$ cannot have a dominant vertex.

Theorem 2.3 Let V be an ordered set of vertices and let A be the v-e matrix of $G=(V,\mu,\rho)$ such that every fuzzy vertex of G is dominant. Then A is positive definite. However the converse is not true.

Proof Note that A is a real symmetric matrix. Each fuzzy vertex of G is dominant implies, A is strictly diagonally dominant. Also since every dominant vertex is an active vertex, each diagonal entry is positive. Now A satisfies all the requirements in the hypothesis of Theorem 1.1. Hence A is positive definite.

The converse of the above theorem is not true. That is, there are fuzzy graphs whose v-e matrix is positive definite but not all its fuzzy vertices are dominant.

The example below shows that none of the fuzzy vertices of the fuzzy graph associated with the given v-e matrix is dominant and moreover it is positive definite.

Example 2.4 Consider the following v-e matrix of a fuzzy graph G with respect to the ordered vertex set V.

$$m_V(G) = \begin{bmatrix} 0.4 & 0.4 & 0.4 & 0.4 \\ 0.4 & 0.6 & 0.6 & 0.5 \\ 0.4 & 0.6 & 0.7 & 0.5 \\ 0.4 & 0.5 & 0.5 & 0.5 \end{bmatrix}$$

The eigen values 0.0291, 0.0483, 0.1436, 1.9789 are all positive however none of its fuzzy vertices is dominant.

Theorem 2.4 Let G_1 and G_2 be two fuzzy graphs such that each of their fuzzy vertices are dominant. Then it is not necessary that every fuzzy vertex of G, the Cartesian product of G_1 and G_2 is also dominant.

Proof The proof is by an example.

Let

$$m_{V_1}(G_1) = \begin{bmatrix} 0.5 & 0.2 \\ 0.2 & 0.7 \end{bmatrix}$$

and

$$m_{V_2}(G_2) = \begin{bmatrix} 0.6 & 0.5 \\ 0.5 & 0.8 \end{bmatrix}$$

be the v-e matrix of G_1 and G_2 respectively.

Let $G=G_1 \times G_2$ be the Cartesian product of G_1 and G_2.

$$m_V(G) = \begin{bmatrix} 0.5 & 0.5 & 0.2 & 0 \\ 0.5 & 0.5 & 0 & 0.2 \\ 0.2 & 0 & 0.6 & 0.5 \\ 0 & 0.2 & 0.5 & 0.7 \end{bmatrix}$$

is the v-e matrix of G. Here we see that all the fuzzy vertices of both G_1 and G_2 are dominant but none of the fuzzy vertices of G is dominant.

3 Conclusions

In this paper, we have defined for the first time the notion of v-e matrix of a fuzzy graph and give a list of properties associated with this definition. Secondly we introduce the notion of dominant vertex of a fuzzy graph and prove that the Cartesian product of two dominant fuzzy graphs need not in general be dominant. Most of the situation is illustrated by examples.

References

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