Ultra Scientist Vol. 27(1)A, 77-89 (2015).

Study of Semigroup Semirings Using chain lattices as Semirings

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(Acceptance Date 19th April, 2015)

Abstract

In this paper we study semigroup semirings using chain lattices as semirings. Such study is new and innovative. Condition for these semigroup semirings to contain special elements like zero divisors, units and idempotents are obtained.

Key words: Semigroup semirings, Chain lattices, symmetric semigroup, zero divisors.

1 Introduction

This paper has three sections. Section one is introductory in nature. Section two studies semigroup semirings using finite monoids as semigroups and chain lattices as semirings. Condition for these semigroup semirings to contain special elements like zero divisors etc are obtained. The conclusions are given in the final section.

2 Semigroup Semirings using chain lattices as semirings :

The definition[1,2] of semigroup semiring is made only to make this paper self contained one. Throughout this paper it is assumed all semigroups are finite and contain identity that is they are finite monoids. Condition for these semigroup semirings to contain special elements like zero divisors etc are obtained. For more about semigroup semirings refer[3,4,5].

Definition 2.1: Let S be a finite monoid under product operation $\times$ (or operations other than $+$) and C_n be the chain lattice of finite order. The set

$$C_nS = \left\{ \sum_{i=1}^{n} a_i s_i \,\middle|\, n < \infty;\ s_i \in S,\ a_i \in C_n, +, \times \right\}$$

with two binary operations '+' and $\times$ is defined as the semigroup semilattice or semigroup semiring if the following conditions are satisfied:

i. If $\alpha = \sum_{i=1}^{n} a_i s_i$ and $\beta = \sum_{i=1}^{n} b_i s_i \in C_nS$ then $\alpha = \beta$ if and only if $a_i = b_i$ for $i = 1, 2, \ldots, n$ and $s_i \in S$.

ii. *Let* $\alpha = \sum_{i=1}^{n} a_i s_i$ *and* $\beta = \sum_{i=1}^{n} b_i s_i$ *be in* C_nS *then*

$$\alpha+\beta = \sum_{i=1}^{n} a_i s_i + \sum_{i=1}^{n} b_i s_i = \sum_{i=1}^{n} (a_i + b_i) s_i$$

$$= \sum_{i=1}^{p} (a_i \cup b_i) s_i$$

is in C_nS.

iii. $\alpha\times\beta = \sum_{i=1}^{n} a_i s_i \times \sum_{j=1}^{m} b_i s_j = \sum_k a_i \times b_j s_k$; $s_k = s_i \times s_j \in S$

$$= \sum_k (a_i \cap b_j) s_k$$

$= \sum_k y_k s_k$; $y_k \in C_n$ *is in* C_nS. *(k runs over finite number).*

iv. $a_i s_i = s_i a_i$ *for all* $a_i \in C_n$ *and* $s_i \in S$.

v. $1.a_i = a_i.1 = a_i$ *for all* $a_i \in C_n$; *1 the identity element of the semigroup of S.*

vi. $1.s_i = s_i.1 = s_i$ *for all* $s_i \in S$ *and* $1 \in C_n$ *is the greatest element of* C_n.

vii. $0.s_i = s_i.0 = 0$ *for all* $s_i \in S$ *and* $0 \in C_n$.

viii. *If* $0 \in S$ *then* $a_i.0 = 0.a_i$ *for all* $a_i \in C_n$.

ix. $\alpha\times(\beta + \gamma) = \alpha\times\beta + \alpha\times\gamma$ *for all* $\alpha, \beta, \gamma \in C_nS$.

First this will be illustrated by some examples.

Example 2.1: Let $C_9 = 0 < a_7 < a_6 < \ldots < a_1 < 1$ be the chain lattice of order *9*. $S = \{Z_{10}, \times\}$ be the semigroup. C_9S be the semigroup semiring of *S* over the chain lattice C_9. C_9-S has zero divisors and $o(C_9S) < \infty$.
For take $x=a_85$ and $y=(a_24+a_38+a_42) \in C_9S$.

$$x+y = a_85 + a_24 + a_38 + a_42 \in C_9S.$$
$$x\times y = (a_85) \times (a_24 + a_38 + a_42)$$
$$= (a_8 \cap a_2)(5\times 4) + (a_8 \cap a_3)(5\times 8) + (a_8\cap a_4)(5\times 2)$$
$$= a_8\times 0 + a_8\ 0 + a_8\times 0$$
$$= 0.$$

Thus C_9S has zero divisors since the semigroup semiring has zero divisors C_9S is not a semifield. Further as *S* is a commutative so is C_9S.

Example 2.2: Let $C_{12} = 0 < a_{10} < a_9 < \ldots < a_1 < 1$ be a chain lattice of order 12. $S = \{Z_{13}, \times\}$ be the semigroup of order 13. $C_{12}S$ be the semigroup semiring of finite order which is commutative.

Clearly if

$x = a_110 + a_35 + a_23 + a_67 + a_{10}$ and $y = a_15 + a_73 + a_54 + a_6 \in C_{12}S$.

$x + y = (a_110 + a_35 + a_23 + a_67 + a_{10}) + (a_15 + a_73 + a_54 + a_6)$

$= a_110 + (a_3 \cup a_1)5 + (a_2 \cup a_7)3 + a_67 + a_54 + a_{10} \cup a_6$

$= a_110 + a_15 + a_23 + a_67 + a_54 + a_6$ is in $C_{10}S$.

Let

$a = a_710 + a_52 + a_8$ and $b = a_59 + a_611 + a_212 + a_1 \in C_{12}S$;

$$\begin{aligned} a \times b &= (a_7 10 + a_5 2 + a_8) \times (a_5 9 + a_6 11 + a_2 12 + a_1) \\ &= (a_7 \cap a_2)\ 10 \times 12 + (a_5 \cap a_2)\ 2 \times 12 + (a_8 \text{-}\cap a_2)\ 1 \times 12 + \\ &\quad (a_7 \cap a_{\text{-}5})\ 10 \times 9 + (a_5 \cap a_5)\ 2 \times 9 + (a_8 \cap a_5)\ 1 \times 9 + (a_7 \cap a_6)\ 10 \times 11 + (a_5 \cap a_6) \\ &\quad 2 \times 11 + (a_8 \cap a_6)\ 11 + (a_7 \cap a_1)\ 10 \times 1 + (a_5 \cap a_1)\ 2 \times 1 + (a_8 \cap a_1) \\ &= a_7 3 + a_5 11 + a_8 12 + a_7 12 + a_5 5 + a_8 9 + a_7 6 + a_6 9 + a_8 11 + a_7 10 + a_5 2 + a_8 \\ &= a_7 3 + (a_8 \cup a_5)11 + (a_8 \cup a_7)12 + a_5 5 + (a_8 \cup a_6)9 + a_7 6 + a_7 10 + a_5 2 + a_8 \\ &= a_7 3 + a_5 11 + a_7 12 + a_5 5 + a_6 9 + a_7 6 + a_7 10 + a_5 2 + a_8. \end{aligned}$$

This is the way product is performed. Clearly $a + b = 0$ is impossible in $C_{12}S$. Thus $C_{12}S$ is a semifield as $a \times b \neq 0$ for any $a, b \in C_{12}S$.

Example 2.3: Let C_{11} be the chain lattice of order 11. Let $S = \{Z_6, \times\}$ be the semigroup of order 6. $C_{11}S$ be the semigroup semiring of finite order. Clearly $C_{11}S$ is commutative but is not a semifield.

For take $x = a_9 3 \in C_{11}S$.
Clearly
$x \times x = a_9 3 \times a_9 3 = (a_9 \cap a_9)\ (3 \times 3) = a_9 3$.

Thus x is an idempotent element of $C_{11}S$.

Consider $y = a_3 4 \in C_{11}S$;
$y \times y = a_3 4 \times a_3 4 = (a_3 \cap a_3)\ (4 \times 4) = a_3 4 = y$.

Thus y is also an idempotent of $C_{11}S$.

Let $a = (a_{10} 3 + a_5 4) \in C_{11}S$

$$\begin{aligned} a \times a &= (a_{10} 3 + a_5 4) \times (a_{10} 3 \times a_5 4) \\ &= (a_{10} \cap a_{10})\ 3 \times 3 + (a_5 \cap a_{10})\ 4 \times 3 + \\ &\quad (a_{10} \cap a_5)\ 3 \times 4 + (a_5 \cap a_{5\text{-}})\ 4 \times 4 \\ &= a_{10} 3 + a_5 4 = a. \end{aligned}$$

Thus a is also an idempotent of $C_{11}S$. $C_{11}S$ has zero divisors.

For take

$$p = a_4 2 + a_5 4 \text{ and } q = a_5 3 \in C_{11}S.$$

$$\begin{aligned} p \times q &= (a_4 2 + a_5 4) \times a_5 3 \\ &= (a_4 \cap a_5)\ 2 \times 3 + (a_5 \cap a_5)\ 4 \times 3 \\ &= 0, \end{aligned}$$

so $C_{11}S$ has non trivial zero divisors. Hence $C_{11}S$ is not a semifield.

Next consider semigroup semiring which is non commutative[6].

Example 2.4: Let $C_{16} = 0 < a_{14} < a_{13} < \ldots < a_2 < a_1 < 1$ be a chain lattice of order 16 and $S = S(4)$ be the symmetric semigroup of order 4^4. Let $C_{16}S$ be the semigroup semiring of finite order. Clearly $C_{16}S$ is non commutative as $S(4)$ is a non commutative semigroup.

Now how sum and product operations are performed on $C_{16}S$ is described briefly.

Let

$$x = a_4\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 1 & 4 & 3\end{pmatrix} + a_{10}\begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 1 & 4 & 2\end{pmatrix} + a_9\begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 1 & 1 & 3\end{pmatrix} + a_6$$

and

$$y = a_{10}\begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 3 & 4 & 1\end{pmatrix} + a_2\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 1 & 4 & 3\end{pmatrix} + a_7\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 2 & 1 & 4\end{pmatrix} + a_{10} \in C_{16\text{-}}S.$$

$$\begin{aligned}
x + y &= (a_4 \cup a_2)\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 1 & 4 & 3\end{pmatrix} + a_{10}\begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 1 & 4 & 2\end{pmatrix}\\
&\quad + a_9\begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 1 & 1 & 3\end{pmatrix} + a_{10}\begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 3 & 4 & 1\end{pmatrix}\\
&\quad + a_7\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 2 & 1 & 4\end{pmatrix} + a_6 \cup a_{10}\\
&= a_2\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 1 & 4 & 3\end{pmatrix} + a_{10}\begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 1 & 4 & 2\end{pmatrix} + a_9\begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 1 & 1 & 3\end{pmatrix} +\\
&\quad a_{10}\begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 3 & 4 & 1\end{pmatrix} + a_7\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 2 & 1 & 4\end{pmatrix} + a_6 \in C_{16}S.
\end{aligned}$$

$$\begin{aligned}
x \times y &= \left[a_4\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 1 & 4 & 3\end{pmatrix} + a_{10}\begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 1 & 4 & 2\end{pmatrix} + a_9\begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 1 & 1 & 3\end{pmatrix} + a_6\right] \times\\
&\quad \left[a_{10}\begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 3 & 4 & 1\end{pmatrix} + a_2\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 1 & 4 & 3\end{pmatrix} + a_7\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 2 & 1 & 4\end{pmatrix} + a_{10}\right]\\
&= (a_4 \cap a_{10})\left[\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 1 & 4 & 3\end{pmatrix} \circ \begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 3 & 4 & 1\end{pmatrix}\right]\\
&\quad + (a_4 \cap a_2)\left[\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 1 & 4 & 3\end{pmatrix} \circ \begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 1 & 4 & 3\end{pmatrix}\right]
\end{aligned}$$

$$+ \ (a_4 \cap a_7) \left[\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 1 & 4 \end{pmatrix} \right]$$

$$+ \ (a_4 \cap a_{10}) \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

$$+ \ (a_{10} \cap a_{10}) \left[\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 1 \end{pmatrix} \right]$$

$$+ \ (a_{10} \cap a_2) \left[\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \right]$$

$$+ \ (a_{10} \cap a_7) \left[\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 1 & 4 \end{pmatrix} \right]$$

$$+ \ (a_{10} \cap a_{10}) \left[\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} \right]$$

$$+ \ (a_9 \cap a_{10}) \left[\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 1 \end{pmatrix} \right]$$

$$+ \ (a_9 \cap a_2) \left[\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \right]$$

$$+ \ (a_9 \cap a_7) \left[\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 1 & 4 \end{pmatrix} \right]$$

$$+ \ (a_9 \cap a_{10}) \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 3 \end{pmatrix} + (a_6 \cap a_{10}) \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 1 \end{pmatrix}$$

$$+ \ (a_6 \cap a_2) \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} + (a_6 \cap a_7) \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 1 & 4 \end{pmatrix} + a_6 \cap a_{10}.$$

$$= \quad a_{10} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 1 & 4 \end{pmatrix} + a_4 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} + a_7 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 4 & 1 \end{pmatrix}$$

$$+ a_{10}\begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix} + a_{10}\begin{pmatrix}1&2&3&4\\4&1&1&3\end{pmatrix} + a_{10}\begin{pmatrix}1&2&3&4\\4&2&3&1\end{pmatrix}$$

$$+ a_{10}\begin{pmatrix}1&2&3&4\\1&2&4&2\end{pmatrix} + a_{10}\begin{pmatrix}1&2&3&4\\3&1&4&2\end{pmatrix} + a_{10}\begin{pmatrix}1&2&3&4\\1&1&1&4\end{pmatrix}$$

$$+ a_9\begin{pmatrix}1&2&3&4\\2&2&2&4\end{pmatrix} + a_9\begin{pmatrix}1&2&3&4\\2&2&2&1\end{pmatrix} + a_{10}\begin{pmatrix}1&2&3&4\\1&1&1&3\end{pmatrix}$$

$$+ a_{10}\begin{pmatrix}1&2&3&4\\1&3&4&1\end{pmatrix} + a_6\begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix} + a_7\begin{pmatrix}1&2&3&4\\2&2&1&4\end{pmatrix}$$

$$+ a_{10} \in C_{16}S.$$

Thus + and × are performed.

Consider

$$y \times x = [a_{10}\begin{pmatrix}1&2&3&4\\1&3&4&1\end{pmatrix} + a_2\begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix} + a_7\begin{pmatrix}1&2&3&4\\2&2&1&4\end{pmatrix}$$

$$+ a_{10}] \times [a_4\begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix} + a_{10}\begin{pmatrix}1&2&3&4\\3&1&4&2\end{pmatrix} +$$

$$a_9\begin{pmatrix}1&2&3&4\\1&1&1&3\end{pmatrix} + a_6]$$

$$= (a_{10} \cap a_4)\left[\begin{pmatrix}1&2&3&4\\1&3&4&1\end{pmatrix} \circ \begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix}\right]$$

$$+ (a_2 \cap a_4)\left[\begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix} \circ \begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix}\right]$$

$$+ (a_7 \cap a_4)\left[\begin{pmatrix}1&2&3&4\\2&2&1&4\end{pmatrix} \circ \begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix}\right]$$

$$+ (a_{10} \cap a_4)\begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix}$$

$$+ \; (a_{10} \cap a_{10}) \left[\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} \right]$$

$$+ \; (a_{2} \cap a_{10}) \left[\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} \right]$$

$$+ \; (a_{7} \cap a_{10}) \left[\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 1 & 4 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} \right]$$

$$+ \; (a_{10} \cap a_{10}) \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$$

$$+ \; (a_{10} \cap a_{9}) \left[\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 3 \end{pmatrix} \right]$$

$$+ \; (a_{2} \cap a_{9}) \left[\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 3 \end{pmatrix} \right]$$

$$+ \; (a_{7} \cap a_{9}) \left[\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 1 & 4 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 3 \end{pmatrix} \right]$$

$$+ \; (a_{10} \cap a_{9}) \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 3 \end{pmatrix} + (a_{10} \cap a_{6}) \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 1 \end{pmatrix}$$

$$+ \; (a_{2} \cap a_{6}) \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} + (a_{7} \cap a_{6}) \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 1 & 4 \end{pmatrix}$$

$$+ \; a_{10} \cap a_{6}$$

$$= \quad a_{10} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 2 \end{pmatrix} + a_{4} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} + a_{7} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \end{pmatrix}$$

$$+ \; a_{10} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} + a_{10} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 3 \end{pmatrix}$$

$$+ a_{10} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix} + a_{10} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 3 & 2 \end{pmatrix} + a_{10} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$$

$$+ a_{10} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 3 & 1 \end{pmatrix} + a_{9} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 3 & 1 \end{pmatrix} + a_{9} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 3 \end{pmatrix}$$

$$+ a_{10} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 3 \end{pmatrix} + a_{10} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 1 \end{pmatrix} + a_{6} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

$$+ a_{7} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 1 & 4 \end{pmatrix} + a_{10}.$$

Clearly $x \times y \neq y \times x$; thus $C_{16}S(4)$ is a non commutative semigroup semiring.

However $C_{16}S(4)$ has no zero divisors but has units as well as idempotents. For take

$$x = a_7 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \in C_{16}S(4);$$

$$x \times x = a_7 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \times a_7 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} = a_7 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} = x.$$

Thus x is an idempotent of $C_{16}S(4)$. Thus $C_{16}S(4)$ is only a semigroup semiring which is a semidivison ring.

In view of this the following result is important:

Proposition 2.1: Let $C_nS(m)$ be the semigroup semiring of the symmetric semigroup $S(m)$ over the chain lattice C_n.

i. $C_nS(m)$ has idempotents.

ii. $C_nS(m)$ is only a semidivision ring.

Proof: Follows from the simple fact

$$x = a_i \begin{pmatrix} 1 & 2 & 3 & \dots & m \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix} \in C_nS(m)$$

is such that $x \times x = x$. Thus $C_nS(m)$ has idempotents. Since $S(m)$ has no zero divisors and $S(m)$ is non commutative semigroup; $C_nS(m)$ is a semidivision ring.

It is an important and an interesting observation to see that the existence of idempotent in a semiring does not imply the existence of a zero divisor a marked difference between rings and semirings.

Next the properties of semigroups $S = \{Z_n, \times\}$ are characterized.

Proposition 2.2: Let $S = \{Z_n, \times\}$ be the semigroup (n a prime) and C_m be the chain lattice of order m. The semigroup semiring C_mS is a semifield.

Proof: Follows from the fact S has no zero divisors as n is a prime; hence C_mS is a semifield; for always $a + b = 0$ if and only if $a = b = 0$ for all $a, b \in C_mS$ and $a.b = 0$ is not possible.

Proposition 2.3: Let C_m be the chain lattice $0 < m - 2 < m - 1 < \ldots < m_2 < m_1 < 1$ and $S = \{Z_n, \times\}$ be the semigroup. C_mS be the semigroup semiring of the semigroup S over the chain lattice C_m. C_mS is not a semifield if and only if Z_n has zero divisors.

Proof: When Z_n has zero divisor clearly C_mZ_n has zero divisors so C_mZ_m is not a semifield. If C_mZ_n has zero divisors since C_m is a chain lattice only zero divisors are contributed by Z_n. This is true from proposition 2.2. Hence the result.

Next consider the matrix semigroup using the natural product $\times_n$. For the notion of natural product in matrices refer[6].

Example 2.5: Let $C_8 = 0 < a_6 < a_5 < \ldots < a_2 < a_1 < 1$ be the chain lattice of order 8 and $S = \{(a_1, a_2, a_3) \mid a_i \in Z_7, 1 \leq i \leq 3, \times\}$ be the matrix semigroup. C_8S is the semigroup semiring. C_8S has zero divisors and units. (1, 1, 1) is the unit of C_8S. C_8S is not a semifield.

Example 2.6: Let

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle| \; a_i \in Z_{17}, 1 \leq i \leq 5, \times_n \right\}$$

be the column matrix semigroup under the natural product $\times_n$. $C_{10} = 0 < a_8 < a_7 < \ldots < a_2 < a_1 < 1$ be the chain lattice of order 10. $C_{10}S$ be the semigroup semiring. $C_{10}S$ has zero divisors.

All the idempotents are only of the form

$$P = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle| \; a_i \in \{0, 1\}; 1 \leq i \leq 5, \times_n \right\} \subseteq S.$$

However $C_{10}S$ has units and the identity element is $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

$x = \begin{bmatrix} 9 \\ 2 \\ 16 \\ 3 \\ 6 \end{bmatrix}$ and $y = \begin{bmatrix} 2 \\ 9 \\ 16 \\ 6 \\ 3 \end{bmatrix} \in C_{10}S$ is such that $x \times_n y = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

Let

$a = \begin{bmatrix} 0 \\ 6 \\ 0 \\ 7 \\ 9 \end{bmatrix}$ and $b = \begin{bmatrix} 7 \\ 0 \\ 8 \\ 0 \\ 0 \end{bmatrix} \in C_{10}S$. $a \times_n b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

is a zero divisor.

Apart from this let

$$\alpha = a_3 \begin{bmatrix} 0 \\ 3 \\ 0 \\ 4 \\ 0 \end{bmatrix} + a_7 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 7 \\ 0 \end{bmatrix} + a_8 \begin{bmatrix} 0 \\ 5 \\ 0 \\ 0 \\ 0 \end{bmatrix} + a_5 \begin{bmatrix} 0 \\ 7 \\ 0 \\ 12 \\ 0 \end{bmatrix}$$

and

$$\beta = a_7 \begin{bmatrix} 4 \\ 0 \\ 7 \\ 0 \\ 6 \end{bmatrix} + a_2 \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 4 \end{bmatrix} + a_1 \begin{bmatrix} 0 \\ 0 \\ 6 \\ 0 \\ 12 \end{bmatrix} \in C_{10}S.$$

$$\alpha \times_n \beta = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus $C_{10}S$ has zero divisors even though the semigroup $S = \{Z_{17}, \times\}$ has no zero divisors.

Example 2.7: Let

$$M = \left\{ \left(\begin{matrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{matrix} \right) \middle| \; a_i \in Z_{13};\ 1 \leq i \leq 12, \times_n \right\}$$

be the matrix semigroup under product $\times_n$. Let $C_2 = \{0, 1\}$ be the chain lattice of order two. C_2M be the semigroup semiring. C_2M has zero divisors however the idempotents are from the subset

$$P = \left\{ \left(\begin{matrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{matrix} \right) \middle| \; a_i \in \{0, 1\};\ 1 \leq i \leq 12, \times_n \right\} \subseteq M.$$

P is the collection of all idempotents from M.

Thus C_2M has idempotents though Z_{13} has no idempotents or zero divisors.

Thus matrix semigroup S under the natural product paves way for several idempotents, units and zero divisors even if S is built using Z_p; p a prime.

Finally this leads to the following result:

Proposition 2.4: Let S = {collection of all $m \times n$ matrices with entries from Z_t; $\times_n$} be the matrix semigroup and $C_s = \{0 < a_{s-2} < a_{s-3} < \ldots < a_2 < a_1 < 1\}$ be the chain lattice of order s. C_sS be the semigroup semiring of S over C_s.

i. C_sS has zero divisors, units and idempotents.

ii. P = {$m \times n$ matrices with entries from {0, 1}} if t is a prime is the only subset of idempotents of S.

iii. Q = {$m \times n$ matrices with entries from {0, 1, collection of all idempotents from Z_t}; if t is a non prime} $\subseteq S$ is the only collection of idempotents of S.

Proof: Follows from the fact S has

zero divisors, units and idempotents. Further (ii) and (iii) can be verified to be true.
This will be illustrated by an example.

Example 2.8: Let

$$S=\left\{\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{pmatrix} \middle| \, a_i \in Z_6; 1 \le i \le 8, \times_n\right\}$$

be the matrix semigroup under the natural product $\times_n$.

$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ is the unit element of S.

Let $C_3 = \{0 < a_1 < 1\}$ be the chain lattice of order three; C_3S be the semigroup semiring of S over C_3. The units of S are

$$M = \left\{\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 \\ 5 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 5 & 1 & 1 \end{pmatrix}, \ldots, \begin{pmatrix} 5 & 5 & 5 & 5 \\ 5 & 5 & 5 & 5 \end{pmatrix}\right\}.$$

That is

$$P = \left\{\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{pmatrix} \middle| \, a_i \in \{1, 5\}\right\} \subseteq S$$

alone are units of S.

Further every $x \in P$ is such that

$$x^2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

P is also a subsemigroup of S. Consider

$$M=\left\{\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{pmatrix} \middle| \, a_i \in \{0, 1, 3, 4\}; 1 \le i \le 8, \times_n\right\} \subseteq S$$

is the collection of all idempotents in S.

Let

$$p = a_1 \begin{pmatrix} 1 & 0 & 3 & 4 \\ 4 & 0 & 1 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 4 & 3 \\ 3 & 4 & 1 & 1 \end{pmatrix} \in C_3S.$$

$$p^2 = \left[a \begin{pmatrix} 1 & 0 & 3 & 4 \\ 4 & 0 & 1 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 4 & 3 \\ 3 & 4 & 1 & 1 \end{pmatrix} \right]^2$$

$$= a \cap a \left[\begin{pmatrix} 1 & 0 & 3 & 4 \\ 4 & 0 & 1 & 3 \end{pmatrix} \times_n \begin{pmatrix} 1 & 0 & 3 & 4 \\ 4 & 0 & 1 & 3 \end{pmatrix} \right]$$

$$+ \begin{pmatrix} 1 & 1 & 4 & 3 \\ 3 & 4 & 1 & 1 \end{pmatrix} \times_n \begin{pmatrix} 1 & 1 & 4 & 3 \\ 3 & 4 & 1 & 1 \end{pmatrix}$$

$$+ (a_1 \cap 1) \begin{pmatrix} 1 & 0 & 3 & 4 \\ 4 & 0 & 1 & 3 \end{pmatrix} \times_n \begin{pmatrix} 1 & 1 & 4 & 3 \\ 3 & 4 & 1 & 1 \end{pmatrix}$$

$$+ (1 \cap a_1) \begin{pmatrix} 1 & 1 & 4 & 3 \\ 3 & 4 & 1 & 1 \end{pmatrix} \times_n \begin{pmatrix} 1 & 0 & 3 & 4 \\ 4 & 0 & 1 & 3 \end{pmatrix}$$

$$= a \begin{pmatrix} 1 & 0 & 3 & 4 \\ 4 & 0 & 1 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 4 & 3 \\ 3 & 4 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

$$\neq p.$$

Thus in general p is not an idempotent of C_3S though

$$\begin{pmatrix} 1 & 0 & 3 & 4 \\ 4 & 0 & 1 & 3 \end{pmatrix} \text{and} \begin{pmatrix} 1 & 1 & 4 & 3 \\ 3 & 4 & 1 & 1 \end{pmatrix}$$

are idempotents further in this case

$$\begin{pmatrix} 1 & 0 & 3 & 4 \\ 4 & 0 & 1 & 3 \end{pmatrix} \times_n \begin{pmatrix} 1 & 1 & 4 & 3 \\ 3 & 4 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

is also an idempotent. Thus in this case M is an idempotent subsemigroup of S.

This may not in general be true for all matrix semigroup built using Z_n.

This is proved by the following examples:

Example 2.9: Let

$$W=\left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix} \middle| \, a_i \in Z_{12}, 1 \le i \le 8, \times_n\right\}$$

be the matrix semigroup under the natural product $\times_n$.

$$P=\left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix} \middle| \, a_i \in \{0, 1, 4, 9\}; 1 \le i \le 8, \times_n\right\} \subseteq W$$

is a subsemigroup of W. P is an idempotent subsemigroup of W. However if $C_{20}W$ is a semigroup semiring and $C_{20}P$ is a subsemiring. But clearly $C_{20}P$ has elements which are not idempotents of $C_{20}W$.

Next the study of idempotent semigroups under $\cup$ (or $\cap$) is analysed first by examples.

Example 2.10: Let $S = \{1, a_1', a_2', \ldots, a_8', \phi, \cap / a_i' \cap a_j' = \phi$ *if* $i \neq j$ $1 \cap a'_1 = a_i$, $a_i' \cap a_i' = a_i'$; $1 \leq i, j \leq 8\}$ be the idempotent semigroup or a semilattice of order 10. $C_{10} = \{0 < a_8 < a_7 < \ldots < a_2 < a_1 < 1\}$ be the chain lattice of order 10. $C_{10}S$ be the semigroup semiring of S over C_{10}.

Let

$$\alpha = a_3 a_1' + a_6 a_3' + a_5.$$

$$\begin{aligned}\alpha^2 &= a_3 a_1' + a_6 a_3' + a_5 + (a_3 \cap a_5) a_1' + (a_6 \cap a_5) a_3' \\ &\quad + (a_3 \cap a_6)(a_1' \cap a_3') + (a_5 \cap a_3) a_1' + (a_5 \cap a_6) a_3' \\ &\quad + (a_6 \cap a_3)(a_3' \cap a_1') \\ &= a_3 + a_6 a_3' + a_5\end{aligned}$$

is an idempotent of $C_{10}S$. This has both zero divisors and idempotents.

It is to be noted that one can in the above definition put $\phi = 0$ without loss of generality. If ϕ is used define $a_i\phi = \phi$ and $\phi.0 = 0$ and ϕ is assumed to be the zero of C_nS.

Example 2.11: Let S be any idempotent semigroup with 1 and 0 such that $s_i \cap s_i = s_i$ *and* $s_i \cap s_j = 0$, $s_i \cap 1 = s_i$; $i \neq j$. C_n *any chain lattice. The semigroup semiring* C_nS *is such that every* $x \in C_nS$; $x^2 = x$. *For take*

$$x = \sum_{i=1}^{n} a_i s_i;\ s_i \in S \setminus \{0, 1\} \text{ and } a_i \in C_n,$$

$$\begin{aligned} x \times x &= \{\sum (a_i \cap a_i)\, s_i \cap s_i + \sum (a_i \cap a_j)\, s_i \cap s_j\} \\ &= \sum_{i=1}^{n} a_i s_i + 0.\end{aligned}$$

Hence the claim. Let

$x = 1 + a_1s_1 + a_3s_2 \in C_nS$.

$x^2 = 1 + a_1s_1 + a_3s_2 + a_1s_1 + a_3s_2 + (a_1 \cap a_3)(s_1 \cap s_2)$
$= x$.

If 1 is replaced by a_4 that is let

$y = a_4 + a_1s_1 + a_3s_2$

$$\begin{aligned} y^2 &= a_4 + a_1s_1 + a_3s_2 + (a_4 \cap a_1)s_1 + (a_4 \cap a_3)\, s_2 \\ &\quad + (a_4 \cap a_1)s_1 \\ &\quad + (a_4 \cap a_3)s_2 + (a_1 \cap a_3)\,(s_1 \cap s_2) + (a_3 \cap a_1) \\ &\quad (s_1 \cap s_2) \\ &= a_4 + a_1s_1 + a_3s_2 \\ &= y.\end{aligned}$$

Thus every element is an idempotent.

In view of this one has the following result in case of idempotent semigroup under $\cap$ of the special form described in Example 2.11.

Proposition 2.5: Let $S = \{1, a_1', a_2', \ldots, a_m', 0 \mid a_i \cap a_j = 0$ *if* $i \neq j$ $a_i \cap a_i = a_i$, $1 \cap a_i' = a_i'$ $0 \cap a_j' = 0$; $1 \leq i, j \leq m\}$ *be the idempotent semigroup.* C_n *be any chain lattice.* C_nS *be the semigroup semiring of the semigroup S over the semiring* C_n. *Every* $\alpha \in C_nS$ *is an idempotent of* C_nS.

Proof: Let $\alpha = \sum_{i=1}^{t} a_i a_i' \in C_nS$.

Clearly $\alpha^2 = \alpha$ in C_nS; hence the claim (using $a'_i \cap a'_j = 0$ if $i \neq j$ and $a_i \cap a_j = a_i$ if $i < j$ or a_j if $j < i$ and $a_i \cup a_j = a_i$ if $i > j$ and a_j if $i < j$).

3. Conclusion

In this paper semigroup semirings using specifically chain lattices as semirings is studied. It is proved semigroup semiring $C_nS(m)$ is semidivision ring. Finally conditions for semigroup semirings C_nS to contain zero divisors, units and idempotents are obtained.

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