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On $\overset{0}{g}$S*- Homeomorphism In Topological Spaces

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Abstract

In this paper we introduce a new class of closed maps namely $\overset{0}{g}$s-closed maps also introduce a new class of homeomorphisms called $\overset{0}{g}$s*-homeomorphisms. Further we show that the set of all $\overset{0}{g}$s*-homeomorphisms form a group under the operation composition of maps.

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1. Introduction

The notion homeomorphism plays an important role in topology. A homeomorphism between two topological spaces X and Y is a bijective map $f : X \to Y$ when both f and f^{-1} are continuous. Manoj and Shikha[4] in 2008 introduced the concept of $\overset{0}{g}$s closed sets in topological spaces. In the present paper we first introduce a new class of closed maps namely $\overset{0}{g}$s-closed maps and then introduce and study $\overset{0}{g}$s*-homeomorphisms in a topological space. We also prove that the set of all $\overset{0}{g}$s*-homeomorphisms forms a group under the operation composition of functions.

2. Preliminaries :

Throughout this paper (X, τ), (Y, σ) and (Z, η) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of space (X, τ) the cl(A), int(A) and A^c denote the closure of A, the interior of A and the complement of A in X respectively.

We recall the following definitions:

Definition 2.1 : A subset A of a topological space (X, τ) is called semi-open [1] (resp. semi-closed[1]) if A $\subseteq$ cl(int(A)) (resp. int(cl(A)) $\subseteq$ A).

The semi-closure[3] of a subset A of X, denoted by scl(A) is defined to be the intersection of all semi-closed sets containing A.

Definition 2.2 : A subset A of a topological space (X, τ) is called g-closed[2] (resp. g*-closed [3], $\overset{0}{g}$s-closed [4]) set if cl(A) $\subseteq$ U (resp. cl(A) $\subseteq$ U, scl(A) $\subseteq$ U) whenever A $\subseteq$ U and U is open (resp. g- open, g*-open) set in (X, τ).

Definition 2.3: A function f : (X, τ) $\rightarrow$ (Y, σ) is called s-continuous[4] (resp. $\overset{0}{g}$s –irresolute[4]) if the inverse image of every σ-closed (resp. $\overset{0}{g}$s-closed) set in Y is $\overset{0}{g}$s-closed (resp. $\overset{0}{g}$s-closed) set in X.

3 *S-Closed Maps :*

In this section we introduce the following definitions.

Definition 3.1: Let (X, τ) be a topological space and A $\subseteq$ X. We define the $\overset{0}{g}$s-closure of A (briefly $\overset{0}{g}$s-cl(A)) to be the intersection of all $\overset{0}{g}$s-closed sets containing A. In symbols, $\overset{0}{g}$s-cl(A) = $\cap${B: A $\subseteq$ B and B $\in$ $\overset{0}{g}$ semi-closure in X.

Theorem 3.2: Let (X, τ) be a topoio-gical space and A $\subseteq$ X. The following properties are hold :

(i) $\overset{0}{g}$s-cl(A) is the smallest $\overset{0}{g}$s-closed set containing A.

(ii) A is $\overset{0}{g}$s-closed iff $\overset{0}{g}$s-cl(A) = A.

Theorem 3.3: For any two subsets A and B of (X, τ),

(i) If A $\subseteq$ B, then $\overset{0}{g}$s-cl(A) $\subseteq$ $\overset{0}{g}$s-cl(B).

(ii) $\overset{0}{g}$s-cl(A $\cap$ B) $\subseteq$ $\overset{0}{g}$s-cl(A $\cap$ B).

Theorem 3.4: If B $\subseteq$ A $\subseteq$ X, B is a $\overset{0}{g}$s-closed set relative to A and A is open and $\overset{0}{g}$s-closed in (X, τ). Then B is $\overset{0}{g}$s-closed in (X, τ).

Corollary 3.5: If A is a $\overset{0}{g}$s-closed set and B is closed set then A $\cap$ B is a $\overset{0}{g}$s-closed set.

Definition 3.6: Let (X, τ) be a topological space and A $\subseteq$ X. We define the $\overset{0}{g}$s-interior of A (briefly $\overset{0}{g}$s-int(A)) to be the union of all $\overset{0}{g}$s-open sets contained in A.

Lemma 3.7 : For any A $\subseteq$ X, Int(A) $\subseteq$ $\overset{0}{g}$s-int(A) $\subseteq$ A.

Proof : Since every open set is $\overset{0}{g}$s-open so proof is obvious.

Definition 3.8 : A map f : (X, τ) $\rightarrow$ (Y, σ) is called $\overset{0}{g}$s-closed (resp. $\overset{0}{g}$s-open) map if f(A) is $\overset{0}{g}$s-closed (resp. $\overset{0}{g}$s-open) set in (Y, σ) for every closed (resp. open) set A of (Y, σ).

Theorem 3.9 : A map f : (X, τ) $\rightarrow$ (Y,

σ) is $\overset{0}{g}$s-closed iff $\overset{0}{g}$s-cl(f(A)) $\subseteq$ f(cl(A)) for every subset A of (X, τ).

Proof : Follows from theorem (3.2) and theorem (3.3).

Theorem 3.10 : A map f : (X, τ) $\rightarrow$ (Y, σ) is $\overset{0}{g}$s-closed iff for each subset A of (Y, σ) and for each open set U containing $f^{-1}(A)$ there exists a $\overset{0}{g}$s-open set V of (Y, σ) such that A $\subseteq$ V and $f^{-1}(V) \subseteq$ U.

Remark 3.11 : The following example shows that the composition of two $\overset{0}{g}$s-closed maps need not be $\overset{0}{g}$s-closed map.

Example 3.12: Let X = {a, b, c}, τ = {ϕ, {a}, {b}, {a, b}, {a, c}, X}, σ = {ϕ, {a}, {a, b}, X} and η = {ϕ, {a, b}, X}. Define f : (X, τ) $\rightarrow$ (X, σ) by identity mapping and g : (X, σ) $\rightarrow$ (X, η) by identity mapping then f and g both are $\overset{0}{g}$s-closed maps but their composition gof : (X, τ) $\rightarrow$ (Z, η) is not a $\overset{0}{g}$s-closed map.

Theorem 3.13: If f : (X, τ) $\rightarrow$ (Y, σ) be a closed map and g : (Y, σ) $\rightarrow$ (Z, η) be a $\overset{0}{g}$s-closed map then their composition gof : (X, τ) $\rightarrow$ (Z, η) is $\overset{0}{g}$s-closed map.

Theorem 3.14: If f : (X, τ) $\rightarrow$ (Y, σ) be a s-closed map and g : (Y, σ) $\rightarrow$ (Z, η) be a closed map, then their composition gof : (X, τ) $\rightarrow$ (Z, η) need not be $\overset{0}{g}$s-closed map.

The following example supports the above theorem.

Example 3.15: Let X = Y = Z = {a, b, c}, τ={ϕ, {a}, {b, c}, X}, σ = {ϕ, {a, c}, Y} and η = {ϕ, {b}, {c}, {b, c}, {a, c}, Z}. Define f : (X, τ) $\rightarrow$ (Y, σ) by f(a) = b, f(b) = b, f(c) = c and g : (Y, σ) $\rightarrow$ (Z, η) by identity mapping. Then f is $\overset{0}{g}$s-closed map and g is closed map but their composition gof : (X, τ) $\rightarrow$ (Z, η) is not a $\overset{0}{g}$s-closed map.

Theorem 3.16: Let f : (X, τ) $\rightarrow$ (Y, σ) and g : (Y, σ) $\rightarrow$ (Z, η) be two mappings such that their composition gof : (X, τ) $\rightarrow$ (Z, η) be a $\overset{0}{g}$-closed map then the following are true

(i) If f is continuous and serjective, then g is $\overset{0}{g}$s-closed map.

(ii) If g is $\overset{0}{g}$s-irresolute and injective, then f is $\overset{0}{g}$s-closed map.

Theorem3.17: Let f_A be the restriction of a map f : (X, τ) $\rightarrow$ (Y, σ) to a subset A of (X, τ) then

(i) If f : (X, τ) $\rightarrow$ (Y, σ) is $\overset{0}{g}$s-closed and A is a closed subset of (X, τ), then f_A : (A, τ_A) $\rightarrow$ (Y, σ) is $\overset{0}{g}$s-closed.

(ii) If f : (X, τ) $\rightarrow$ (Y, σ) is $\overset{0}{g}$s-closed (resp. closed) and A = $f^{-1}(B)$ for some closed (resp. $\overset{0}{g}$s-closed) set B of (Y, σ) then f_A : (A, τ_A) $\rightarrow$ (Y, σ) is $\overset{0}{g}$s-closed.

Proof: (i) Let B be a closed set of A. Then B = A $\cap$ C for some closed set C of (X, τ) and so B is closed in (X, τ). By hypothesis, f(B) is $\overset{0}{g}$s-closed in (Y, σ). But f(B) = $f_A(B)$ and so f_A is a $\overset{0}{g}$s-closed map.

(ii) Let D be a closed set of A. Then D = A $\cap$ E for some closed set E in (X, τ). Now $f_A(D) = f(D) = f(A \cap E) = f(f^{-1}(B) \cap E) = B \cap f(E)$. Since f is s-closed, f(E) is $\overset{0}{g}$s-closed and so $B \cap f(E)$ is $\overset{0}{g}$s-closed in (Y, σ) by corollary (3.05). Thus f_A is s-closed map.

Theorem 3.18: For any bijective $f : (X, \tau) \to (Y, \sigma)$ the following statements are equivalent.

(i) $f^{-1} : (Y, \sigma) \to (X, \tau)$ is $\overset{0}{g}$s-continuous.

(ii) f is $\overset{0}{g}$s-open map and

(iii) f is $\overset{0}{g}$s-closed map.

Theorem 3.19: If $f : (X, \tau) \to (Y, \sigma)$ be a $\overset{0}{g}$s-open mapping then for a subset A of (X, τ), $f(\text{int}(A)) \subseteq \overset{0}{g}\text{s-int}(f(A))$.

Theorem 3.20 : A map $f : (X, \tau) \to (Y, \sigma)$ is $\overset{0}{g}$s-open iff for any subset B of (Y, σ) and for any closed set A containing $f^{-1}(B)$, there exists a $\overset{0}{g}$s-closed set C of (Y, σ) containing B such that $f^{-1}(C) \subseteq A$.

Proof: Similar to theorem (3.10).

Corollary 3.21: A map $f : (X, \tau) \to (Y, \sigma)$ is s-open iff $f^{-1}(\text{s-cl}(A)) \subseteq \text{cl}(f^{-1}(A))$ for every subset A of (Y, σ).

Definition 3.22: A map $f : (X, \tau) \to (Y, \sigma)$ is said to be a $\overset{0}{g}$s*-closed (resp. $\overset{0}{g}$s*-open) map if the image f(A) is $\overset{0}{g}$s-closed (resp. $\overset{0}{g}$s-open) set in (Y, σ) for every $\overset{0}{g}$s-closed (resp. $\overset{0}{g}$s-open) set A in (X, τ).

Theorem 3.23: Every $\overset{0}{g}$s*-closed map is $\overset{0}{g}$s-closed map.

The converse is not true in general as seen from the following example.

Example 3.24: Let X = Y = {a, b, c}, $\tau = \{\phi, \{a\}, \{b, c\}, X\}$ and $\sigma = \{\phi, \{b\}, \{c\}, \{b, c\}, \{a, c\}, Y\}$. Define $f : (X, \tau) \to (Y, \sigma)$ by f(a) = b, f(b) = a, f(c) = c. Then f is $\overset{0}{g}$s-closed map but not $\overset{0}{g}$s*-closed map.

Theorem 3.25: A map $f : (X, \tau) \to (Y, \sigma)$ is $\overset{0}{g}$s*-closed iff $\overset{0}{g}\text{s-cl}(f(A)) \subseteq f(\overset{0}{g}\text{s-cl}(A))$ for every subset A of (X, τ).

Proof: Similar to theorem (3.9).

Theorem 3.26: For any bijection $f : (X, \tau) \to (Y, \sigma)$ the following are equivalent :

(i) $f^{-1} : (Y, \sigma) \to (X, \tau)$ is $\overset{0}{g}$s-irresolute,

(ii) f is a $\overset{0}{g}$s*-open map and

(iii) f is a $\overset{0}{g}$s*-closed map.

Proof: Similar to theorem (3.18).

4. $\overset{0}{g}$s-Homeomorphisms :*

In this section we introduce the following definitions.

Definition 4.1: A bijection $f : (X, \tau) \to (Y, \sigma)$ is said to be $\overset{0}{g}$s*-homeomorphisms if both f and f^{-1} are $\overset{0}{g}$-irresolute.

We denote the family of all $\overset{0}{g}$s*-

homeomorphism of a topological space (X, τ) onto itself by $\overset{0}{g}$s*h (X, τ).

Theorem 4.2: If $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ are $\overset{0}{g}$s*-homeomorphisms, then their composition gof : $(X, \tau) \to (Z, \eta)$ is also $\overset{0}{g}$s*-homeomorphism.

Theorem 4.3: The set $\overset{0}{g}$s*-h(X, τ) is a group under the composition of maps.

Theorem 4.4: If $f : (X, \tau) \to (Y, \sigma)$ be a $\overset{0}{g}$s*-homeomorphism, then f induces an isomorphism from the group $\overset{0}{g}$s*-h(X, τ) onto the group $\overset{0}{g}$s*-h(Y, σ).

Proof: Define ψ_f : $\overset{0}{g}$s*-h$(X, \tau) \to$ $\overset{0}{g}$s*-h(Y, σ) by $\psi_f(h)$ = $fohof^{-1}$ for every h $\in \overset{0}{g}$s*-h(X, τ). Then ψ_f is a bijection. Further, for all $h_1, h_2 \in \overset{0}{g}$s*-h$(X, \tau)$, $\psi_f(h_1 o h_2) = fo(h_1 o h_2) o f^{-1} = (foh_1 o f^{-1})\ o\ (foh_2 o f^{-1}) = \psi_f(h_1)\ o\ \psi_f(h_2)$ so ψ_f is a homeomorphism and hence it is an isomorphism induced by f.

Theorem 4.5: $\overset{0}{g}$s*-homeomorphism is an equivalence relation in the collection of all topological spaces.

Proof: Follows from theorem (4.2).

Theorem 4.6: If $f : (X, \tau) \to (Y, \sigma)$ is a $\overset{0}{g}$s*-homeomorphism, then $\overset{0}{g}$s-cl$(f^{-1}(A)) \subseteq f^{-1}(\overset{0}{g}$s-cl((B)) for all $A \subseteq Y$.

Corollary 4.7: If $f : (X, \tau) \to (Y, \sigma)$ is a $\overset{0}{g}$s*-homeomorphism, then $\overset{0}{g}$s-cl(f(A)) = f($\overset{0}{g}$s-cl((A)) for all $A \subseteq X$.

Corollary 4.8: If $f : (X, \tau) \to (Y, \sigma)$ is $\overset{0}{g}$s*-homeomorphism, then f($\overset{0}{g}$s-int(A)) = $\overset{0}{g}$s-int(f(A)) for all $A \subseteq X$.

Proof: Follows from corollary (4.7).

Corollary 4.9: If $f : (X, \tau) \to (Y, \sigma)$ is a $\overset{0}{g}$s*-homeomorphism, then $f^{-1}(\overset{0}{g}$s-int(A)) = $\overset{0}{g}$s-int$(f^{-1}(A))$ for all $A \subseteq Y$.

Proof: Follows from corollary (4.8).

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