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Generalized Sasakian Space form Admitting Semi-Symmetric Connection

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Abstract

In this paper, we study $\tilde{\tau}$ curvature tensor in generalised Sasakian space forms with respect to semi-symmetric metric connection $\tilde{M}(f_1, f_2, f_3)$. We obtain scalar curvature of $\tilde{\tau}$ -Ricci semi-symmetric $\tilde{M}(f_1, f_2, f_3)$. Further we prove that φ-$\tilde{\tau}$ flat $\tilde{M}(f_1, f_2, f_3)$ and $\tilde{\tau}$ - semi-symmetric $\tilde{M}(f_1, f_2, f_3)$ are η -Einstein.

1. Introduction

An n -dimensional differentiable manifold M is called an almost metric manifold if its structural group can be reduced to U(n) × I or equivalently, there is an almost contact structure (φ, ξ, η) consisting of a (1,1) tensor field φ, a vector field ξ and a 1- form η satisfying

$$\phi^2 X = -X + \eta(X)\xi, \ \phi\xi = 0, \tag{1.1}$$

$$\eta(\xi) = 1, \ \ \eta \circ \phi = 0 \tag{1.2}$$

Let g be a Riemannian metric compatible with (φ, ξ, η). Then we have

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{1.3}$$

$$g(\phi X, Y) = -g(X, \phi Y), \quad g(X, \xi) = \eta(X),$$

for any vector fields X, Y on M [2]. Also we have

$$(\nabla_X \phi)Y = -\eta(Y)\phi X - g(X, \phi Y)\xi.$$

Alegre *et.al*[1] introduced and studied the generalised Sasakian space forms. An almost contact metric manifold (M, φ, ξ, η, g) is said to be a generalised Sasakian space form if there exist differentiable functions f_1, f_2, f_3, such that curvature tensor R of M is given by

$$R(X,Y)Z = f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\}$$

$$+f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}. \quad (1.4)$$

In a generalised Sasakian space form the following results hold

$$S(Y,Z) = \big((n-1)f_1 + 3f_2 - f_3\big)g(Y,Z) - (3f_2 + (n-2)f_3)\eta(Y)\eta(Z), \quad (1.5)$$

$$QY = [(n-1)f_1 + 3f_2 - f_3]Y - [3f_2 + (n-2)f_3]\eta(Y)\xi, \quad (1.6)$$

$$S(Y,\xi) = (n-1)(f_1 - f_3)\eta(Y), \quad (1.7)$$

$$Q\xi = (n-1)(f_1 - f_3)\xi, \quad (1.8)$$

$$r = n(n-1)f_1 + 3(n-1)f_2 - 2(n-1)f_3, \quad (1.9)$$

$$R(X,Y)\xi = (f_1 - f_3)(\eta(Y)X - \eta(X)Y), \quad (1.10)$$

$$R(\xi,X)Y = (f_1 - f_3)(g(X,Y)\xi - \eta(Y)X), \quad (1.11)$$

$$\eta(R(X,Y)Z) = (f_1 - f_3)\big(g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\big).$$

Let $\tilde{\nabla}$ be a linear connection in an n dimensional differentiable manifold M. The torsion tensor $\tilde{T}$ is given by $\tilde{T}(X,Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y]$. [The connection $\tilde{\nabla}$ is symmetric if its torsion tensor vanishes, otherwise it is non-symmetric.] If there is a Riemannian metric g in M such that $\tilde{\nabla} g = 0$, then the connection $\tilde{\nabla}$ is a metric connection, otherwise it is non-metric[3]. It is known that a linear connection is symmetric and metric if and only if it is a Levi-civita connection. A linear connection in a Riemannian manifold is said to be a semi-symmetric connection if its torsion tensor $\tilde{\nabla}$ is of the form[4,5]

$$\tilde{T}(X,Y) = \pi(Y)X - \pi(X)Y, \quad (1.12)$$

where the 1-form π is defined by $\pi(X) = g(X, \rho)$ and ρ is a vector field.

In an almost contact metric manifold, a semi-symmetric connection is defined by $\tilde{T}(X,Y) = \eta(Y)X - \eta(X)Y$ with ξ as associated vector field (i,e $g(X,\xi) = \eta(X)$). A relation between the semi-symmetric connection $\tilde{\nabla}$ and the Levi-civita connection ∇ of M has been obtained by K.Yano and it is given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X,Y)\xi. \quad (1.13)$$

In this paper we study generalized Sasakian space form $M(f_1, f_2, f_3)$ admitting semi-symmetric connection and we prove $\tilde{\tau}$-Ricci semi-symmetric, φ-$\tilde{\tau}$ flat and $\tilde{\tau}$-semi-symmetric $M(f_1, f_2, f_3)$ are η -Einstein.

2. Generalized Sasakian Space Form Admitting Semi-Symmetric Connection :

Let R and $\tilde{R}$ denote the Riemannian curvature of M with respect to Levi-civita connection ∇ and semi-symmetric connection $\tilde{\nabla}$ respectively[6].

By definition of $\tilde{R}$ and equation (1.13), we have

$$\tilde{R}(X,Y)Z = R(X,Y)Z - \alpha(Y,Z)X + \alpha(X,Z)Y - g(Y,Z)FX + g(X,Z)FY, \quad (2.1)$$

$$\alpha(Y,Z) = (\nabla_Y \eta)Z - \eta(Y)\eta(Z) + \frac{1}{2}\eta(\xi)g(Y,Z) \quad (2.2)$$

Or

$$\alpha(Y,Z) = (\nabla_Y \eta)Z - \frac{1}{2}g(Y,Z)$$

F be a tensor field of type (1,1) is given by

$$g(FY,Z) = \alpha(Y,Z) \quad (2.3)$$

for any vector fields Y, Z. Contracting (2.1) with W, we get

$$\tilde{R}(X,Y,Z,W) = R(X,Y,Z,W) - \alpha(Y,Z)g(X,W) + \alpha(X,Z)g(Y,W) - g(Y,Z)\alpha(X,W) + g(X,Z)\alpha(Y,W). \quad (2.4)$$

Taking W = ξ in (2.4) and using (2.2), we obtain

$$\eta\left(\tilde{R}(X,Y)Z\right) = (f_1 - f_3)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] + g(\phi Y,Z)\eta(X) - \eta(Y)g(\phi X,Z). \quad (2.5)$$

Using (2.1) and (1.4), we obtain

$$\tilde{R}(\xi,Y)\xi = (f_1 - f_3)(\eta(X)\xi - X) - \frac{1}{2}X + \frac{1}{2}\eta(X)\xi + FX - \eta(X)F\xi\,, \quad (2.6)$$

$$\tilde{R}(X,Y)\xi = \left(f_1 - f_3 + \frac{1}{2}\right)(\eta(Y)X - \eta(X)Y) + \eta(X)FY - \eta(Y)FX, \quad (2.7)$$

$$\tilde{S}(X,W) = \left[(n-1)f_1 + 3f_2 - f_3 - n + 2\right]g(X,W) - -[3f_2 + (n-2)f_3 - n + 2]\,\eta(X)\eta(W) + (n-2)g(\phi X,W), \quad (2.8)$$

$$\tilde{S}(X,\xi) = (n-1)(f_1 - f_3)\eta(X), \quad (2.9)$$

$$\tilde{S}(\phi X,\phi W) = \tilde{S}(X,W) - (n-1)(f_1 - f_3)\eta(X)\eta(W). \quad (2.10)$$

In an almost contact manifold M, the curvature tensor $\tilde{\tau}$ with respect to semi-symmetric connection $\tilde{\nabla}$ is given by

$$\tilde{\tau}(X,Y)Z = a_0\tilde{R}(X,Y)Z + a_1\tilde{S}(Y,Z)X + a_2\tilde{S}(X,Z)Y + a_3\tilde{S}(X,Y)Z + a_4 g(Y,Z)\tilde{Q}X + a_5 g(X,Z)\tilde{Q}Y + a_6 g(X,Y)\tilde{Q}Z + a_7\tilde{r}[g(Y,Z)X - g(X,Z)Y] \quad (2.11)$$

for X, Y, Z ∈ T_PM., where a_0,..., a_7 are some smooth functions on M and $\tilde{R}$, $\tilde{S}$, $\tilde{Q}$ are the Riemannian curvature tensor, Ricci tensor and Ricci operator with respect to the connection $\tilde{\nabla}$ respectively.

3. $\tilde{\tau}$-Ricci Semi-Symmetric, Φ-$\tilde{\tau}$ Flat and $\tilde{\tau}$- Semi-Symmetric Generalized Sasakian Space Forms Admitting To Semi-Symmetric Metric Connection :

A generalised Sasakian space form $M(f_1, f_2, f_3)$ is said to be $\tilde{\tau}$-Ricci semi - symmetric if $\tilde{\tau}.\tilde{S} = 0$. The condition $\tilde{\tau}.\tilde{S} = 0$ implies that

$$\tilde{S}(\tilde{\tau}(U,V)X, Y) + \tilde{S}(X, \tilde{\tau}(U,V)Y) = 0.$$

Taking X = Y = ξ in the above equation, we have

$$\tilde{S}\left(\tilde{\tau}(U,V)\xi,\xi\right) \quad (3.1)$$

Put Z = ξ in (2.9) and using (2.11), we get

$$\tilde{\tau}(X,Y)\xi = [a_0\left(f_1 - f_3 + \frac{1}{2}\right) + a_1(n-1)(f_1 - f_3) + a_4((n-1)f_1 + 3f_2 - f_3 - n + 2\,) + a_7\tilde{r}]\eta(Y)X + \left[-a_0\left(f_1 - f_3 + \frac{1}{2}\right) + a_2(n-1)(f_1 - f_3) + a_5\big((n-1)f_1 + 3f_2 - f_3 - n + 2\,\big) - a_7\tilde{r}\right]\eta(X)Y + a_3\tilde{S}(X,Y)\xi - [a_4(3f_2 + (n-2)f_3 - n + 2)\eta(Y)(X) - a_5(3f_2 + (n-2)f_3 - n + 2)\eta(Y)\eta(X) + a_6(n-1)f_1 -$$

$$f_3 g(X,Y)]\xi + a_0[\eta(X)QY - \eta(Y)QX] + a_4(n-2)\eta(Y)\phi X + a_5(n-2)\eta(X)\phi Y \tag{3.2}$$

In view of (3.1) and (3.2), we have

$$\big[a_1(n-1)^2(f_1-f_3)^2 + a_4\big((n-1)f_1 + 3f_2 - f_3 - n + 2\big)(n-1)(f_1-f_3) + a_2(n-1)^2(f_1-f_3)^2 + a_5\big((n-1)f_1 + 3f_2 - f_3 - n + 2\big)(n-1)(f_1-f_3) - a_4(3f_2 + (n-2)f_3 - n + 2)(n-1)(f_1-f_3\big) - a_5(3f_2 + (n-2)f_3 - n + 2)(n-1)(f_1 - f_3)]\eta(X)\eta(Y) + a_3\tilde{S}(X,Y)(n-1)(f_1-f_3) + a_6(n-1)^2(f_1-f_3)^2 g(X,Y) = 0\ .$$

or equivalently

$$\tilde{S}(X,Y) = \left[-\frac{a_6}{a_3}\left((n-1)(f_1-f_3)\right)\right] g(X,Y) + \left[\frac{a_4+a_5}{a_3}\left((n-1)(f_1-f_3)(-a_1-a_2-1)\right)\right]\eta(X)\eta(Y). \tag{3.3}$$

Let e_i be an orthonormal basis of vector fields in M.

Setting X = Y = e_i in (3.3) and summing over i = 1,..,n, we get the scalar curvature

$$\tilde{r} = \frac{(n-1)(f_1-f_3)}{a_3}[-na_6 - (a_4+a_5)(1+a_1+a_2)] \tag{3.4}$$

Thus we can state the following

Theorem 3.1. A generalized Sasakian space form with respect to semi-symmetric metric connection satisfying $\tilde{\tau}.\ \tilde{S} = 0$ is η – Einstein and the scalar curvature is given by (3.4).

A generalized Sasakian space form $M(f_1, f_2, f_3)$ with respect to the semi-symmetric connection $\varphi - \tilde{\tau}$ flat if

$$\phi^2\tilde{\tau}(\phi X, \phi Y, \phi Z, \phi W) = 0. \tag{3.5}$$

If $M(f_1, f_2, f_3)$ *is* $\varphi - \tilde{\tau}$ flat, then

$$g(\tilde{\tau}(\phi X, \phi Y)\phi Z, \phi W) = 0 \tag{3.6}$$

$\forall$ X, Y, Z, W $\in T_P M$.

Contracting (3.1) with respect to W and replacing X, Y, Z, W by φX, φY, φZ, φW respect-ively, we obtain

$$\tilde{\tau}(\phi X, \phi Y, \phi Z, \phi W) = a_0\tilde{R}(\phi X, \phi Y, \phi Z, \phi W) + a_1\tilde{S}(\phi Y, \phi Z)g(\phi X, \phi W) + a_2\tilde{S}(\phi X, \phi Z)g(\phi Y, \phi W) + a_3\tilde{S}(\phi X, \phi Y)g(\phi Z, \phi W) + a_4 g(\phi Y, \phi Z)\tilde{S}(\phi X, \phi W) + a_5 g(\phi X, \phi Z)\tilde{S}(\phi Y, \phi W) + a_6 g(\phi X, \phi Y)\tilde{S}(\phi Z, \phi W) + a_7\tilde{r}[g(\phi Y, \phi Z)g(\phi X, \phi W) - g(\phi X, \phi Z)g(\phi Y, \phi W)] \tag{3.7}$$

$\forall$ X, Y, Z, W $\in T_P M$. If $\{e_1, e_2, .. e_{n-1}, \xi\}$ be local orthonormal basis of the vector fields in M then $\{\phi e_1, \phi e_2, .. \phi e_{n-1}, \xi\}$ is also a local orthonormal basis[1-4].

Using (1.5), (1.11) and (2.10), we get

$$\sum_{i=1}^{n-1}\tilde{R}(\varphi e_i, \varphi Y, \varphi Z, \varphi e_i) = \tilde{S}(Y,Z) - [(2n-3)f_1$$

$$+(4-n)f_3+\frac{1}{2}]\eta\ (Y)\eta(Z-[(n+2)f_1+3f_3\frac{1}{2}]g(Y,Z) \quad (3.8)$$

By noting

$$\sum_{i=1}^{n-1}\tilde{S}(\varphi e_i,\varphi e_i)=\tilde{r}-(n-1)(f_1-f_3) \quad (3.9)$$

and using (3.7) and (3.8), we obtain

$$-\tilde{S}(Y,Z)-\left[(2n-3)f_1+(4-n)f_3+\frac{1}{2}\right]\eta(Y)\eta(Z)$$
$$-\left[(n+2)f_1+3f_3+\frac{1}{2}\right]g(Y,Z)$$
$$=(a_1(n-1)+a_2+a_3+a_5+a_6)\tilde{S}(\phi Y,\phi Z)$$
$$+[a_1\tilde{r}-(n-1)(f_1-f_3)+a_7\tilde{r}(n-2)]g(Y,Z) \quad (3.10)$$

Using (1.3) in (2.10), we get

$$\tilde{S}(Y,Z)=Ag(Y,Z)+B\eta(Y)\eta(Z) \quad (3.11)$$

Hence we can state the following

Theorem 3.2. If a generalised Sasakian space form is $\varphi-\tilde{\tau}$ flat with respect to semi-symmetric connection then the manifold is η– Einstein manifold with respect to the semi-symmetric connection.

A Riemannian manifold M is called $\tilde{\tau}$ semi-symmetric if its curvature tensor R satisfies the condition $\tilde{\tau}$ o R = 0. It is interesting to investigate the semi-symmetry of the special Riemannian manifolds. It is known that locally symmetric manifolds are trivially semi-symmetric[5,6].

Let us consider $\tilde{\tau}$- semi- symmetric generalised Sasakian space form admitting a semi-symmetric connection. Then the condition

$$\tilde{R}(X,Y).\tilde{\tau}=0 \quad (3.12)$$

holds on M for every vector fields X,Y.

Contracting (2.11) with respect to ξ and using (2.5), we have

$$\eta(\tilde{\tau}(X,Y)Z)=(f_1-f_3+a_7\tilde{r})(g(Y,Z)\eta(X)$$
$$-g(X,Y)\eta(Y)+g(\phi Y,Z)\eta(X)$$
$$-g(\phi X,Z)\eta(Y)+((n-1)f_1+3f_2$$
$$-f_3-n+2)(a_1g(Y,Z)\eta(X)$$
$$+a_2g(X,Z)\eta(Y)+a_3g(X,Y)\eta(Z)$$
$$-(3f_2+(n-2)f_3-n+2)(a_1+a_2$$
$$+a_3)\eta(X)\eta(Y)\eta(Z)$$
$$+(n-1)(f_1-f_3)\big(a_4g(Y,Z)\eta(X)$$
$$+a_5g(X,Z)\eta(Y)+a_6g(X,Y)\eta(Z)\big) \quad (3.13)$$

Taking Z = ξ, (3.11) we get

$$\eta(\tilde{\tau}(X,Y)\xi)=[(n-1)(f_1-f_3)(a_1+a_2+a_4$$
$$+a_5)-a_3(3f_2+(n-2)f_3-n+2)]\eta(X)\eta(Y)$$
$$+\left[(n-1)(f_1-f_3)(a_6)+a_3\big((n-1)f_1+3f_2\right.$$
$$\left.-f_3-n+2\big)\right]g(X,Y) \quad (3.14)$$

Taking X = ξ in (3.11) we get

$$\eta(\tilde{\tau}(\xi,Y)Z)=[-(f_1-f_3+a_7\tilde{r})+(a_2+a_3$$
$$+a_5+a_6)(n-1)(f_1-f_3)-a_1(3f_2+(n-2)f_3$$
$$-n+2)]\ \eta(Y)\eta(Z)+[(f_1-f_3+a_7\tilde{r}+a_4(n$$
$$-1)(f_1-f_3)\quad+a_1\big((n-1)f_1+3f_2-f_3-n$$
$$+2\big)]g(Y,Z)+g(\phi Y,Z) \quad (3.15)$$

The condition (3.10) implies that

$$\tilde{R}(X,Y)\ \tilde{\tau}(U,V)Z-\ \tilde{\tau}\big(\tilde{R}(X,Y)U,V\big)Z-\tilde{\tau}\big(U,$$
$$\tilde{R}(X,Y)V\big)Z-\tilde{\tau}\left((U,V)(\tilde{R}(X,Y)\right)Z=0 \quad (3.16)$$

Therefore

$$g(\tilde{R}(\xi,Y)\,\tilde{\tau}(U,V)Z,\xi)-g\big(\tilde{\tau}\big(\tilde{R}(\xi,Y)U,V\big)Z,\xi\big)$$
$$-\,g\big(\tilde{\tau}\big(U,\tilde{R}(\xi,Y)V\big)Z,\xi\big)$$
$$-g(\tilde{\tau}(U,V)(R(\xi,Y)Z,\xi)=0\ .$$

From which it follows that

$$(f_1-f_3)\tilde{\tau}(U,V,Z,Y)-(f_1-f_3)\eta(Y)\eta(\tilde{\tau}(U,V)Z)$$
$$+\,g(\phi Y,\tilde{\tau}(U,V)Z)-$$
$$\eta(\tilde{\tau}(R(\xi,Y)U,V)Z)-\eta(\tilde{\tau}(U,R(\xi,Y)V)Z)$$
$$-\,\eta(\tilde{\tau}(U,V)R(\xi,Y)Z=0 \qquad (3.17)$$

If $\{e_1,e_2,...e_{n-1},\xi\}$ be local orthonormal basis of the vector fields in M then from (3.15), we get

$$\sum_{i=1}^{n-1}\tilde{\tau}(e_i,V,Z,e_i)=(a_0+a_1+a_2+a_3$$
$$+a_5+a_6)\tilde{S}(V,Z)-[-(f_1-f_3+a_7+$$
$$(a_2+a_3)\big((n-1)f_1+3f_2-f_3-n+2\big)$$
$$+(a_1+a_2+a_3)(3f_2+(n-2)f_3-n+2)+$$
$$(a_5+a_6)(n-1)(f_1-f_3)\eta(V)\eta(Z)+\big[(a_4+a_7n)\tilde{r}$$
$$+(f_1-f_3)+a_1\big((n-1)f_1+3f_2-f_3-n+2\big)$$
$$+a_4(n-1)(f_1-f_3)\big]g(V,Z) \qquad (3.18)$$

$$\sum_{i=1}^{n-1}\tilde{\tau}\,(e_i,V,Z,\phi e_i)=(-a_0+a_7\tilde{r})g(\phi V,Z)$$
$$+(-a_2+a_3-a_5+a_6)\tilde{S}(\phi V,Z)+a_4g(V,Z) \qquad (3.19)$$

Using (2.8), (3.16) and (3.17) in (3.15), we get

$$\tilde{S}(V,Z)=Ag(V,Z)+B\eta(V)\eta(Z)$$

Hence we can state the following

Theorem 3.3. A $\tilde{\tau}$-semi-symmetric generalised Sasakian space form admitting a semi- symmetric connection is η – Einstein.

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