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Groupoid Semirings Which Satisfy Special Identities

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Abstract

In this paper we study groupoid semirings which satisfy special identities. Such study is new and innovative.

Key words: Groupoid semirings, semirings, groupoids, distributive lattices, chain lattice, P-groupoid semiring, Bol semiring, Alternative semiring.

1. Introduction

This paper is organized into three sections. Section one is introductory in nature. Section two defines groupoid semirings which satisfy special identities. Conclusion are provided in the last section.

Throughout this paper groupoids are built using Z_n. These concepts are recalled from[2]. Just for the sake of completeness we recall them.

Definition 1.1: Let $Z_n = \{0, 1, 2, ..., n-1\}$; $n \geq 3$. For a, $b \in Z_n \backslash \{0\}$ define a binary operation $*$ on Z_n as follows; $a * b = ta+ub$ (mod n) where t, u are 2 elements in Z_n here ' + ' is the usual addition of 2 integers and ' ta ' means the product of the two integers t and a. $\{Z_n, (t, u), *\}$ is defined as the groupoid. We denote this groupoid by $\{Z_n, (t, u), *\}$ or in short by Z_n (t, u).

It is interesting to note that for varying t, $u \in Z_n$, we get a collection of groupoids for a fixed integer n.

Example 1.1: Let $Z_7 = \{0, 1, 2, ... , 6\}$. The groupoid Z_7 (3, 4) is given by the following table:

$*$	0	1	2	3	4	5	6
0	0	4	1	5	2	6	3
1	3	0	4	1	5	2	6
2	6	3	0	4	1	5	2
3	2	6	3	0	4	1	5
4	5	2	6	3	0	4	1
5	1	5	2	6	3	0	4
6	4	1	5	2	6	3	0

This is a finite groupoid of order 7, which is non commutative.

Definition 1.2: A groupoid G is said to be a Moufang groupoid if it satisfies the

Moufang identity (xy) (zx) = (x(yz))x for all x, y, z in G.

Example 1.2: Let (G, ∗) be a groupoid given by the following table:

∗	a_0	a_1	a_2	a_3
a_0	a_0	a_2	a_0	a_2
a_1	a_1	a_3	a_1	a_3
a_2	a_2	a_0	a_2	a_0
a_3	a_3	a_1	a_3	a_1

$$(a_1 * a_3) * (a_2 * a_1) = (a_1 * (a_3 * a_2)) * a_1$$
$$(a_1 * a_3) * (a_2 * a_3) = a_3 * a_0 = a_3.$$

Now $(a_1 * (a_3 * a_2))\ a_1 = (a_1 * a_3) * a_1 = a_3 * a_1 = a_1$. Since $(a_1 * a_3)\ (a_2 * a_1) \neq (a_1\ (a_3 * a_2)) * a_1$, we see G is not a Moufang groupoid.

Definition 1.3: A groupoid G is said to be a Bol groupoid if G satisfies the Bol identity ((xy) z) y = x ((yz) y) for all x, y, z in G.

Definition 1.4: A groupoid G is said to be a P-groupoid if (xy) x = x (yx) for all x, y ∈ G.

Definition 1.5: A groupoid G is said to be right alternative if it satisfies the identity (xy) y = x (yy) for all x, y ∈ G. Similarly define G to be left alternative if (xx) y = x (xy) for all x, y ∈ G.

Definition 1.6: A groupoid G is alternative if it is both right and left alternative, simultaneously.

For more about these groupoids and the identities they satisfy refer [2]. Throughout this paper $C_n = 0 < a_1 < \ldots < a_{n-2} < 1$ denotes a chain lattice and L is a distributive lattice. For properties and definitions of semirings. Please refer[1,3].

*Definition 1.7: Let (G, *) be a groupoid of finite order G may or may not contain identity. Let C_n be a chain lattice of order n. Let C_nG = {Collection of all finite formal sums of the form $\sum_i \alpha_i g_i$; i; runs over a finite index; $\alpha_i \in C_n$ and $g_i \in G$}. Define two binary operations + and × on C_nG.*

$$For\ \alpha = \sum_i \alpha_i g_i \ and\ \beta = \sum_i \beta_i g_i \in C_nG$$

$$\alpha + \beta = \sum_i \alpha_i g_i + \sum_i \beta_i g_i$$

$$= \sum_i (\alpha_i + \beta_i) g_i \ (\alpha_i,\ \beta_i \in C_n \ and\ \alpha_i + \beta_i = \alpha_i \cup \beta_i \in C_n)$$

$$= \sum_i (\alpha_i \cup \beta_i) g_i.$$

$$\alpha + 0 = \sum_i \alpha_i g_i + 0 = \sum_i (\alpha_i \cup 0) g_i$$

$$= \sum_i \alpha_i g_i = \alpha.$$

$$\alpha \times \beta = \sum_i \alpha_i g_i \times \sum_i \beta_i g_i$$

$$= \sum_i \gamma_k g_k \text{ where } \gamma_k = \alpha_i \beta_j = \alpha_i \cap \beta_j \in G.\ g_k = g_i * g_j \in G.$$

Clearly $0 \cdot \alpha = 0 = 0\,\alpha$ *for all* $\alpha \in C_n G$, $0 \in C_n$.

As $1 \in C_n$; $1 \cdot g_i \in C_n G$, *thus* $G \subseteq C_n G$; *however as 1 may or may not be present in G.* $C_n \not\subset C_n G$ *if* $1 \notin G$.

Thus $\{C_n G, +, \times\}$ *is defined as the groupoid semiring of the groupoid G over the semiring* C_n.

The following observations are essential.

i. $C_n G$ is a non associative semiring.
ii. $C_n G$ may or may not contain identity element; $C_n G$ has 1 if and only if $1 \in G$.
iii. $C_n G$ is commutative if and only if G is commutative.

Thus $C_n G$ in general is a non associative, non commutative semiring without identity.

This method enables one to obtain semirings which are non associative in a nice way. Since both C_n and G are of finite order the groupoid semiring $C_n G$ is of finite order.

All these groupoid semirings are non associative semirings. For the first time a systematic study of these non associative semirings satisifying special identities is carried out.

2. Groupoid semirings which satisfy special identities :

In this section groupoid semirings which satisfy special identifies like P-semirings, alternative semirings, Bol semirings and Moufang semirings are defined, developed and their properties are given.

Definition 2.1: Let LG be the groupoid semiring of the groupoid G over a distributive lattice L. If for all ,α, $\beta \in LG$; $(\alpha\beta)\alpha = \alpha(\beta\alpha)$ *then LG is defined to be a P-semiring (P- groupoid semiring).*

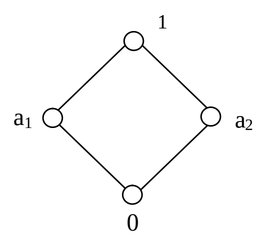
First examples of them will be given.

Example 2.1: Let $G = \{Z_{20}, (5, 0), *\}$ be the groupoid and C_{15} be the chain lattice. $C_{15}G$ be the groupoid semiring. $C_{15}G$ is a P-groupoid semiring or in short P-semiring.

Example 2.2: Let $G = \{Z_6, (4, 0)\ *\}$ be a groupoid. C_7 be the chain lattice. C_7G is the P-groupoid semiring.

Example 2.3: Let $G = \{Z_6, (0, 3), *\}$ be the groupoid.

L =



be the lattice. LG be the groupoid semiring. Let

$$\alpha = (a_1g_2 + a_2g_0 + g_3 + a_1g_4)$$

and

$$\beta = (a_1g_4 + a_2g_5 + a_2g_3 + g_2) \in LG.$$

$(g_0 = 0, g_1 = 1, g_2 = 2, g_3 = 3, g_4 = 4, g_5 = 5)$

$$(\alpha\beta)\alpha = [(a_1g_2+a_2g_0+g_3+ a_1g_4) \times (a_1g_4+a_2g_5+ a_2g_3 + g_2)]\ \alpha$$
$$= (a_1g_0 + 0 + a_1g_0 + a_1g_0 + 0 + a_2g_3 + a_2g_3 + 0 + 0 + a_2g_3 + a_2g_3 + 0 + a_1g_0 + a_2g_0 + g_0 + a_1g_0)\ \alpha$$
$$= (g_0 + a_2g_3)\ (a_1g_2 + a_2g_0 + g_3 + a_1g_4)$$
$$= a_1g_0 + 0 + a_2g_0+a_2g_0+a_2g_3 + g_3 + a_1g_0 + 0$$
$$= g_0 + g_3 \qquad \ldots \text{I}$$

Now

$$\alpha(\beta\alpha) = \alpha[(a_1g_4 + a_2g_5 + a_2g_3 + g_2) \times (a_1g_2 + a_2g_0 + g_3 + a_1g_4)]$$
$$= \alpha\ (a_1g_0+ 0 + 0 + a_1g_0 + 0 + a_2g_0 + a_2g_0 + g_3 + a_1g_0 + 0 + 0 + a_1g_0)$$
$$= (a_1g_2 + a_2g_0 + g_3 + a_1g_4) \times (g_0 + g_3)$$
$$= a_1g_0 + a_2g_0 + g_0+a_1g_0+a_1g_3+a_2g_3+g_3+a_1g_3$$
$$= g_0 + g_3 \qquad \ldots \text{II}$$

I and II are equal. Thus if G is a P-groupoid so is for $\alpha, \beta \in LG$.

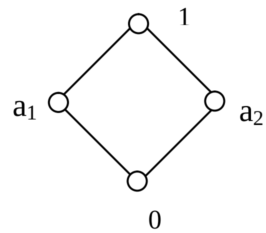
In view of this the following result is true.

Theorem 2.1: Let $G = \{Z_n, (0, t), *\}$ be the groupoid. L be any lattice. LG the groupoid semiring is a P-semiring if and only if $t^2 \equiv t (\text{mod } n)$.

Proof: If for $\alpha, \beta \in LG$; $(\alpha\beta)\alpha = \alpha(\beta\alpha)$ then LG is a P-groupoid semiring. LG is a P groupoid semiring if and only if G is a P-groupoid and that is if and only if $t^2 \equiv t$ (mod n). proved as in [2].

Other types of P-semirings are got using the following P-groupoids.

Example 2.4: Let $G = \{Z_{12}, (5, 5), *\}$ be the groupiod L be any lattice. LG be the groupoid semiring. LG is a P-groupoid semiring, where

L = 

For if $\alpha, \beta \in LG$. $(\alpha \times \beta) \times \alpha = (\alpha\beta)\ \alpha = \alpha\ (\beta\alpha) = \alpha \times (\beta \times \alpha)$.

Take

$$\alpha = a_1g_{10} + a_2g_4 + a_2g_5$$

and

$$\beta = a_1g_0 + g_6 + a_2g_7 \in LG.$$

$$\alpha\times(\beta\times\alpha) = \alpha\ [(a_1g_0 + g_6 + a_2g_7)\times (a_1g_{10} + a_2g_4 + a_2g_5)]$$
$$= \alpha\ [a_1g_2 + a_1g_8 + 0 + 0 + a_2g_2 +a_2g_7 + 0 + a_2g_7 + a_2g_0]$$
$$= (a_1g_{10} + a_2g_4 + a_2g_5) \times (g_2 + a_2g_7 + a_2g_0 + a_1g_8)$$
$$= a_1g_0 + a_2g_6+a_2g_0 + a_2g_{11} + 0 + a_2g_7 + 0 + a_2g_8 + a_2g_1 + a_1g_6 + 0 + 0$$
$$= g_0 + g_6 + a_2g_{11} + a_2g_7 + a_2g_1 + a_2g_8 \qquad \ldots \text{I}$$

$$(\alpha\beta)\alpha = \alpha(\beta\alpha) \text{ as } \alpha\beta = \beta\alpha.$$

Hence LG is a P-groupoid.

In view of this we have the following theorem.

Theorem 2.2: Let $G = \{Z_n, (t, t), *\}$ be the groupoid. L be any distributive lattice.

LG is a P-groupoid semiring.

Proof: Follows from the simple fact G is a P-groupoid. Similar to the proof given in[2].

Next the concept of Bol groupoid semiring is defined in the following.

Definition 2.2: Let LG be a groupoid semiring. LG is a Bol groupoid semiring (Bol Semiring) if and only if $((\alpha\beta)\gamma)\beta = \alpha((\beta\gamma)y)$ for all $\alpha, \beta, \gamma \in LG$.

Some examples of Bol groupoid semiring will be provided in the following.

Example 2.5: Let G = $\{Z_{20}, (4, 5), *\}$ be the Bol groupoid. L = C_9 be the chain lattice and LG be the groupoid semiring. LG is a Bol groupoid semiring.

Let

$$\alpha = a_1g_7 + a_5g_2,\ \beta = a_7g_5 + a_6g_0$$

and

$$\gamma = a_1g_{18} \in LG.$$

Consider

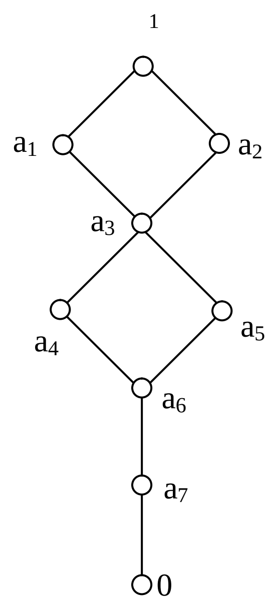
$$((\alpha\times\beta)\times\gamma)\times\beta=(((a_1g_7 + a_5g_2) \times (a_7g_5 + a_6g_0)) \times \gamma) \times \beta$$
$$= ([a_7g_{13} + a_7g_{13} + a_6g_8 + a_6g_8] \times \gamma) \times \beta$$
$$= [(a_7g_{13} + a_6g_8)a_1g_{18}] \times (a_7g_5 + a_6g_0)$$
$$= (a_7g_2 + a_6g_2) \times (a_7g_5 + a_6g_0)$$
$$= a_6g_2 \times (a_7g_5 + a_6g_0)$$
$$= a_7g_{13} + a_6g_8 \qquad \ldots \text{ I}$$

$$\alpha\times[(\beta\times\gamma)\times\beta]=\alpha \times ([(a_7g_5 + a_6g_0) \times a_1g_{18}]\ \beta)$$
$$= \alpha\times([a_7g_{10} + a_6g_{10}] \times (a_7g_5 + a_6g_0))$$
$$= \alpha \times (a_6g_{10} \times (a_7g_5 + a_6g_0))$$
$$= \alpha \times (a_7g_5 + a_6g_0$$
$$= (a_1g_7 + a_5g_2) \times (a_7g_5 + a_6g_0)$$
$$= a_7\ (g_{13}) + a_7g_{13} + a_6g_8 + a_6g_8$$
$$= a_7g_{13} + a_6g_8 \qquad \ldots\text{II}$$

I and II are identical. Thus LG = C_9G is Bol groupoid semiring. (Since G is a Bol groupoid from [2]).

Example 2.6: Let G=$\{Z_{15}, (10, 5); *\}$ be the Bol groupoid.

L =



be the distributive lattice. LG be the groupoid semiring. LG is a Bol groupoid semiring or Bol semiring.

In view of all this the following theorem is proved.

Theorem 2.3: Let G = $\{Z_n, (t, u), *\}$ be a groupoid. L be any distributive lattice. LG be the groupoid semiring. LG is a Bol groupoid semiring if and only if $t^3 = t \pmod n$ and $u^2 = u \pmod n$.That is if and only if G *is a Bol groupoid.*

Proof. Follows from the fact G is a Bol groupoid if and only if $t^3 = t \pmod n$ and $u^2 = u \pmod n$).[2].

Next the concept of alternative, left alternative and right alternative groupoid semirings will be defined.

Definition 2.3: Let LG be a groupoid semiring of a groupoid G over the distributive lattice L. LG is a left alternative groupoid ring if and only if $(xx)y = x(xy)$ for all $x, y \in LG$. LG is a right alternative groupoid semiring if and only if $(xy)y = x(yy)$ for all $x, y \in LG$. If LG is both left and right alternative groupoid semiring then LG is defined as the alternative groupoid semiring.

Examples of these will be provided in the following.

Example 2.7: Let (G, *) be the groupoid given by the following table.

*	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
a_0	a_0	a_6	a_2	a_8	a_4	a_0	a_6	a_2	a_8	a_4
a_1	a_5	a_1	a_7	a_3	a_9	a_5	a_1	a_7	a_3	a_9
a_2	a_0	a_6	a_2	a_8	a_4	a_0	a_6	a_2	a_8	a_4
a_3	a_5	a_1	a_7	a_3	a_9	a_5	a_1	a_7	a_3	a_9
a_4	a_0	a_6	a_2	a_8	a_4	a_0	a_6	a_2	a_8	a_4
a_5	a_5	a_1	a_7	a_3	a_9	a_5	a_1	a_7	a_3	a_9
a_6	a_0	a_6	a_2	a_8	a_4	a_0	a_6	a_2	a_8	a_4
a_7	a_5	a_1	a_7	a_3	a_9	a_5	a_1	a_7	a_3	a_9
a_8	a_0	a_6	a_2	a_8	a_4	a_0	a_6	a_2	a_8	a_4
a_9	a_5	a_1	a_7	a_3	a_9	a_5	a_1	a_7	a_3	a_9

$$(a_5 * a_3) * a_3 = a_3$$
$$a_5 * (a_3 * a_3) = a_3.$$

It is easily verified G is right alternative.

Consider

$$(a_5 * a_5)a_3 = a_5 * a_3 = a_3$$
$$a_5 * (a_5 * a_3) = a_3$$

it is also left alternative. Hence G is an alternative groupoid.

LG is an alternative groupoid semiring for every distributive lattice L.

Example 2.8: Let $G = \{Z_{12}, (0, 9), *\}$ be the groupoid. L be the chain lattice C^{20}. $C_{20}G$ is an alternative groupoid semiring as G is an alternative groupoid[2].

Example 2.9: Let $G=\{Z_{15}, (10, 10), *\}$

be the groupoid L any distributive lattice. LG is the groupoid semiring. LG is an alternative groupoid semiring. G is an alternative groupoid[2].

In view of all these the following theorem is proved.

Theorem 2.4: Let $G = \{Z_n, (0, t), *\}$ *be a groupoid and L a distributive lattice. LG the groupoid semiring is an alternative groupoid semiring if and only if* $t^2 = t(\text{mod } n)$*; that is if and only if G is an alternative groupoid.*

Proof. Follows from the fact G is an alternative groupoid if and only if $t^2 = t$ (mod n). Hence LG is an alternative groupoid semiring if and only if $t^2 = t$ (mod n)[2].

Corollary 2.1: Let $G = \{Z_n, (0, t), *\}$ be the groupoid. L any distributive lattice. LG is not an alternative groupoid semiring if n is a prime.

Proof. If n is a prime clearly $t^2 = t$ (mod n) is not possible.

Using above theorem the result is true.

Example 2.10: Let $G= \{Z_6, (3, 4), *\}$ be a groupoid. L any distributive lattice. LG is an alternative semiring.[2]

Example 2.11: Let $G=\{Z_{12}, (4, 9), *\}$ be a groupoid. L any distributive lattice. LG is an alternative groupoid semiring[2].

In view of this the following theorem is proved.

Theorem 2.5: Let $G = \{Z_n, (t, u), *\}$ *be a groupoid. L any distributive lattice. LG be the groupoid semiring. LG is an alternative groupoid semiring if and only if* $u^2 = u$ *(mod n),* $t^2 = t$ *(mod n) and* $t + u = 1$ *(mod n).*

Proof: LG is an alternative groupoid semiring if and only if G is an alternative groupoid. But G is an alternative groupoid if and only if $u^2 = u$ (mod n), $t^2 = t$ (mod n) and $t + u = 1$ (mod n) [2].

Hence LG is a alternative groupoid semiring if and only if $t^2 = t$ (mod n), $u^2 = u$ (mod n) and $t + u \equiv 1$ (mod n).

Next the notion of Moufang groupoid semiring or Moufang semiring is defined.

Definition 2.4: Let LG be any groupoid semiring of a groupoid G over a distributive lattice L. LG is a Moufang groupoid if and only if $(xy)(zx) = (x(yz))x$ *for all* $x, y, z \in LG$.

First this will be illustrated by the following examples.

Example 2.12: Let $G = \{Z_{15}, (10, 6), *\}$ be the groupoid. L any distributive lattice. LG is a Moufang groupoid as G is a Moufang groupoid.

Example 2.13: Let $G=\{Z_{20}, (16, 5), *\}$ be the groupoid. $L = C_{40}$ be the chain lattice. LG be the groupoid semiring. LG is a Moufang groupoid semiring as G is a Moufang groupoid.

In view of this the following theorems are proved.

*Theorem 2.6: Let G = {Z_n, (t, u), *} be the groupoid. L any distributive lattice. LG be the groupoid semiring. LG is a Moufang groupoid semiring if and only if $t^2 = t \pmod n$, $u^2 = u \pmod n$ and $t + u \equiv 1 \pmod n$.*

Proof: Given LG is a groupoid semiring. G is a Moufang groupoid if and only if $t^2 = t \pmod n$, $u^2 = u \pmod n$ and $t + u \equiv 1 \pmod n$.[2]

Hence LG is a Moufang groupoid semiring as G is a Moufang groupoid. Hence the theorem.

3. Conclusions

In this paper the notion of non associative semirings built using groupoids semirings analysed.

References

1. G. Birkhoff and S. S. Maclane, A Brief Survey of Modern Algebra, New York, The Macmillan and Co. (1965).
2. Vasantha Kandasamy, W.B., Groupoids and Smarandache groupoids, American Research Press, Rehoboth, (2002).
3. Vasantha Kandasamy, W.B., Smarandache Semirings, Semifields and Semivector spaces, American Research Press, Rehoboth (2002).