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On Fuzzy Dot Γ- hypersub-near-algebras

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Abstract

In this paper we introduce Γ-hypernear-algebra and obtain some related properties. Also we introduce fuzzy dot Γ-hypersub-near-algebra and some properties are described.

1.Introduction

The theory of algebraic hyperstructures (or hypersystems) is a well established branch of classical algebraic theory. In the literature, the theory of hyperstructure was first initiated by Marty in 1934 [4] when he defined the hypergroups and began to investigate their properties with applications to groups, rational functions and algebraic functions. Later on, many people have observed that the theory of hyperstructures also has many applications in both pure and applied sciences, It is noted that the study of hypernear-rings is challenging, effecting curiously beautiful results to one who is willing to look for structure where symmetry is not so abundant. After the introduction of the concept of fuzzy sets by Zadeh in 1965 [8], there were many papers devoted to fuzzify the classical mathematics into fuzzy mathematics. A new algebraic system Γ-near-algebra was introduced by T. Srinivas, P. Narashimha Swamy and K. Vijayakumar[6]. Γ- near –algebra is generalizalition of both the concepts of near-algebra and Γ- near-ring. The fuzzy sets and hyperstructures introduced by Zadeh and Marty, respectively, are now extensively applied to many disciplines. The relationships between the fuzzy sets and algebraic hyperstructures have been considered by Corsini, Davvaz, Leoreanu, Zhan, Zahedi and others.

Motivated by these concepts, in this paper we introduce the concept of Γ- hyper near-algebras and obtain some related properties of Γ-hypernear–algebras. In section 4 we introduce fuzzy dot Γ-hypersub-near-algebras and some properties are described.

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2. Preliminaries :

Definition 2.1. A Γ-near-ring is a triple (M, +, Γ), where
(i) Γ is a non-empty set of binary operations such that (M, +, α) is a near-ring for each $\alpha \in \Gamma$, (ii) $x\alpha(y\beta z) = (x\alpha y)\beta z$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

Definition 2.2. A subset A of a Γ-near-ring M is called a *left (resp., right) ideal* of M if (i) *(A, +)* is a normal subgroup of (M, +),

(ii) $u\alpha x \in A$ (resp., $(u + x)\alpha v - u\alpha v \in A$) for all $x \in A$, $\alpha \in F$ and $u, v \in M$.

A subset A of *M* is called an *ideal* of *M* if it is both a left ideal and right ideal.

A *hyperstructure* is a non-empty set *H* together with a mapping o:IxI→ *P*(H)*, where *P*(H)* is the set of all the non-empty subsets of *H*.

Definition 2.3. A *canonical hypergroup* (not necessarily commutative) is an algebraic structure *(H, +)* satisfying the following conditions:

(i) for every $x, y, z \in H$, $x+(y+z)=(x+y)+z$;
(ii) there exists a $0 \in S$ such that $0 + x = x + 0 = x$, for all $x \in H$; (iii) for every $x \in H$, there exists a unique element $x' \in H$ such that $0 \in (x + x') \cap (x' + x)$. (we call the element x' the opposite of x); (iv) $z \in x + y$ implies $y \in -x+ z$ and $x \in z-y$.

Definition 2.4. we recall that a *hypernear-ring* is an algebraic structure (R, +, **.**) satisfying the following axioms:

(1) *(R, +)* is a canonical hypergroup;
(2) with respect to the multiplication, (R, **.**) is a semigroup;
(3) the multiplication is distributive with respect to the hyperoperation '+' on the left hand side, i.e., $x \cdot (y+ z) = x.y + x.z$, for all $x,y,z \in R$.

Definition 2.5. A Γ-hypernear-ring is a triple (M,+,Γ)

(i) Γ is a non-empty set of binary operations such that (M,+,α) is a hypernear-ring for each $\alpha \in \Gamma$.
(ii) $x\,\alpha(y\,\beta\, z)=(x\,\alpha\, y)\beta\, z$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

Definition 2.6. A subset *A* of M is called a left (resp. right) hyperideal of M if it satisfies:

(i) (*A*,+) is a normal subhypergroup of (M,+),
(ii) $u\alpha x \in A$ (resp., $(u + x)\alpha v - u\alpha v \in A$) for all $x \in A$, $\alpha \in \Gamma$ and $u, v \in M$.

A subset A of *M* is called a *hyperideal* of M if it is both a left hyperideal and right hyperideal.

Definition 2.7. Let X be a field and F be a fuzzy set on X. F is called a fuzzy field of X denoted by (F, X) if

(i) $F(x - y) \geq F(x) \wedge F(y), \quad x, y \in X$
(ii) $F(-x)) \geq F(x), \quad x \in X$
(iii) $F(x\, y)) \geq F(x) \wedge F(y), \quad x, y \in X$
(iv) $F(x^{-1}) \geq F(x), \quad x \neq 0 \in X$

3. Γ-hypernear-algebras :

Definition 3.1. Let M be a linear space over a field X and Γ be a non-empty set. Then M is said to be a Γ-hypernear-algebra (left Γ-hypernear -algebra) over a field X if there exists a mapping M x Γ x M→ (the image of (a, α, b) is denoted by a α b satisfying the following conditions:

(i) (a α b) β c = a α (b β c)

(ii) a α (b+ c) = a α b + a α c where + is hyper operatio

(iii) (λ a) α b = λ(a α b) for all a, b, c ϵ M, α, β ϵ Γ and λ ϵ X.

Definition 3.2. A subspace *A* of M is called a left (resp. right) Γ-hyperideal of M if it satisfies:

uαx Є A (resp., (u + *x)*αv — uα*v* Є A) for all *x* Є A, *α* Є Γ and u, *v* Є M.

A subspace A of *M* is called a Γ-*hyperideal* of M if it is both a left Γ-hyperideal and right Γ-hyperideal.

Theorem 3.3. Let I_1 and I_2 be two Γ-hyper ideals of a Γ- hypernear- algebra M over a Field F. Then $I_1 \cap I_2$ is an Γ- hyperideal of M.

Proof. Given that I_1 and I_2 are two Γ-hyper ideals of a Γ- hypernear -algebra M. Then I_1 and I_2 are subspaces of the linear space M. So $I_1 \cap I_2$ is a linear subspace of the linear space M. Let x, y ∈ M, α ∈ Γ, i ∈ $I_1 \cap I_2$. Then i ∈ I_1 and i ∈ I_2. Now for any x, y ∈ M, α ∈ Γ, i ∈ I_1 and since I_1 is an ideal of M, we have (x+i) αy – xαy ∈ I_1, xαi ∈ I_1. Also for any x, y ∈ M, α ∈ Γ, i ∈ I_2 and since I_2 is an ideal of M, we have (x+i) αy – xαy ∈ I_2, xαi ∈ I_2. Thus we obtain (x+i) αy – xαy ∈ $I_1 \cap I_2$ and xαi ∈ $I_1 \cap I_2$. Hence $I_1 \cap I_2$ is an ideal of M.

Theorem 3.4. Let I_1 and I_2 be two left Γ- hyperideals of a Γ-hypernear-algebra M over a ϋeld F. Then $I_1 + I_2$ is a left Γ- hyper ideal of M.

Proof. Given that I_1 and I_2 are two left Γ-hyperideals of a Γ-hypernear-algebra M. Then it is clear that $I_1 + I_2 \subseteq$ M. Let i, j ∈ $I_1 + I_2$. Then $i = i_1 + i_2$ and $j = j_1 + j_2$;$i_1, j_1 \in I_1$; $i_2, j_2 \in I_2$. If a,b ∈ F, then $ai_1 + bj_1 \in I_1$ and $ai_2 + bj_2 \in I_2$. Now, $ai + bj = a(i_1 + i_2) + b(j_1 + j_2) = (ai_1 + bj_1) + (ai_2 + bj_2) \in I_1 + I_2$. Therefore for every a,b ∈ F and i, j ∈ $I_1 + I_2$ we have ai + bj ∈ $I_1 + I_2$. Thus $I_1 + I_2$ is a linear subspace of M over a field F. Let x ∈ M, i ∈ $I_1 + I_2$ and α ∈ Γ. Then for $i = i_1 + i_2$, where $i_1 \in I_1, i_2 \in I_2$, we have $x\alpha i = x\alpha(i_1 + i_2) = x\alpha i_1 + x\alpha\, i_2 \in I_1 + I_2$.

Thus I_1+I_2 is a left Γ-hyperideal of M.

Remark 3.5. Let I_1 and I_2 be two Γ -hyper ideals of a Γ- hyper near-algebra M over a field F. If $i_1 \in I_1$ and $0 \in I_2$, then $i_1 + 0 = i_1 \in I_1 + I_2$. Therefore $I_1 \subseteq I_1 + I_2$. Similarly $I_2 \subseteq I_1 + I_2$. Hence $I_1 \cup I_2 \subseteq I_1 + I_2$.

Definition 3.6. A non-empty subset L of a Γ- hypernear-algebra M over a field X is said to be a sub Γ-hypernear-algebra of M if the following three conditions hold:

i) x-y, k x Є L,

ii) x α y Є L and

iii) (kx) α y =k(xαy) for every x, y Є L, k Є X *and α* Є Γ

Definition 3.7. Let M and M′ be two Γ-hyper near-algebras over a field F. A mapping $\varphi : M \to M'$ is Γ-hypernear- algebra homomorphism if the following three conditions hold: (1) $\varphi(x + y) = \varphi(x) + \varphi(y)$, (2) $\varphi(\lambda x) = \lambda\varphi(x)$, (3) $\varphi(x\alpha y) = \varphi(x)\alpha\varphi(y)$ for every x, y $\in$ M, $\lambda \in X$, $\alpha \in \Gamma$. We say that φ is a Γ-hypernear-algebra isomorphism if φ is one-one, onto and homomorphism.

Theorem 3.8. If $\varphi : M \to M'$ is a Γ hyper near-algebra homomorphism, then the homomorphic image φ(M) is a sub Γ-hyper near-algebra of M′.

Proof. Given that M and M′ are two Γ-hyper near-algebras over a field X, and $\varphi : M \to M'$ is a Γ-hypernear-algebra homomorphism. The homomorphic image of M is φ(M) $= \{x' \in M' : \varphi(x) = x', x \in M\}$. To show φ(M) is a sub Γ- hypernear-algebra of M′, it is sufficient to prove that:

Let $x', y' \in \varphi(M), \alpha \in \Gamma, \lambda \in X$. Then there exists $x, y \in M$ such that $x' = \varphi(x)$, $y' = \varphi(y)$. Since M is a Γ-hypernear-algebra, we get $x - y, \lambda x, x\alpha y \in M$ and $(\lambda x)\alpha y = \lambda(x\alpha y)$. Now: (1) $x' - y' = \varphi(x) - \varphi(y) = \varphi(x - y) \in \varphi(M)$ and $\lambda x' = \lambda\varphi(x) = \varphi(\lambda x) \in \varphi(M)$. (2) $x'\alpha y' = \varphi(x)\alpha\varphi(y) = \varphi(x\alpha y) \in \varphi(M)$. (3) $\lambda(x'\alpha y') = \lambda(\varphi(x)\alpha\varphi(y)) = \lambda(\varphi(x\alpha y)) = \varphi(\lambda(x\alpha y)) = \varphi((\lambda x)\alpha y) = \varphi(\lambda x)\alpha\varphi(y) = (\lambda\varphi(x))\alpha\varphi(y) = (\lambda x')\alpha y'$. Hence φ(M) is a sub Γ-hypernear-algebra of M′.

Definition 3.9. Let M and M′ be two Γ-hyper near-algebras over a field F. Let $\varphi : M \to M'$ be a Γ hyper near-algebra homomorphism. Then the Kernel of φ is denoted by Kerφ and is defined by $\text{Ker}\varphi = \{x \in M : \varphi(x) = 0'\}$, where 0′ is the zero element in M′.

Theorem 3.10. If $\varphi : M \to M'$ is a Γ-hyper near-algebra homomorphism, then Kerφ is an Γ-hyperideal of M.

Proof. Given that M and M′ are two Γ-hypernear-algebras over a field X, and $\varphi : M \to M'$ is a Γ-hypernear-algebra homomorphism. Since $\varphi(0) = 0'$, we get $0 \in \text{Ker}\varphi$. That is Ker φ is a non-empty subset of M. Let $a, b \in X$, $x, y \in M$, $\alpha \in \Gamma$, $i, j \in \text{Ker}\varphi$. Then $\varphi(i) = 0', \varphi(j) = 0'$.

Now: (1) since M itself is a linear space, we get $ai + bj \in M$. Also $\varphi(ai + bj) = \varphi(ai) + \varphi(bj) = a\varphi(i) + b\varphi(j) = a0' + b0' = 0' + 0' = 0'$. Therefore $ai + bj \in \text{Ker}\,\varphi$. Thus Kerφ is a linear subspace of M.

(2) For every $i \in \text{Ker}\varphi$, $\alpha \in \Gamma$, $x \in M$, we have that $x\alpha i \in M$ and $\varphi(x\alpha i) = \varphi(x)\alpha\varphi(i) = \varphi(x)\alpha 0' = 0'$. Therefore $x\alpha i \in \text{Ker}\,\varphi$. Thus Kerφ is a left Γ- hyper ideal of M.

(3) Since $x, y \in M$, $i \in \text{Ker}\varphi$ $\alpha \in \Gamma$ and M is a Γ-hypernear-algebra, we get $(x+i)\alpha y, x\alpha y \in M$. This implies that $(x+i)\alpha y - x\alpha y \in M$(since M is a linear space). Consider $\varphi((x+i)\alpha y - x\alpha y) = \varphi((x+i)\alpha y) - \varphi(x\alpha y) = \varphi(x + i)\alpha\varphi(y) - \varphi(x\alpha y) = (\varphi(x) + \varphi(i))\,\alpha\,\varphi(y) - \varphi(x\alpha y) = (\varphi(x) + 0')\,\alpha\,\varphi(y) - \varphi(x\alpha y) = \varphi(x)\alpha\varphi(y) - \varphi(x\alpha y) = \varphi(x\alpha y) - \varphi(x\alpha y) = 0'$.

Therefore $(x+i)\alpha y - x\alpha y \in \text{Ker}\,\varphi$. Thus Ker φ is a right Γ-hyperideal of M. Hence Ker φ is an Γ-hyperideal of M.

Theorem 3.11. Let M and M′ be two Γ hyper near-algebras over a field F. Let $\varphi : M \to M'$ be a Γ- hypernear-algebra homomorphism. If L is an Γ- hyper ideal of M, then φ(L) is an Γ- hyperideal of φ(M).

Proof. We know that $\varphi(M) = \{x' \in M' : \varphi(x) = x', x \in M\}$ and $\varphi(L) = \{\varphi(y) : y \in L \subseteq M, \varphi(y) = y', y' \in \varphi(M)\}$. Then $\varphi(L) \subseteq \varphi(M)$. Let $a,b \in F; i', j' \in \varphi(L)$. Then $i', j' \in \varphi(M)$ and there exists $i, j \in L$ such that $\varphi(i) = i', \varphi(j) = j'$. Since L is an Γ- hyperideal in M, we get $ai + bj \in L$ for every $a, b \in F$; $i, j \in L$. Now $ai' + bj' = a\,\varphi(i) + b\,\varphi(j) = \varphi(ai) + \varphi(bj) = \varphi(a\,i + bj) \in \varphi(L)$(since $ai + bj \in L$). Therefore for every $a,b \in X$; $i', j' \in \varphi(L), ai' + bj' \in \varphi(L)$. Thus $\varphi(L)$ is a linear subspace of $\varphi(M)$. Let $i' \in \varphi(L), \alpha \in \Gamma, x' \in \varphi(M)$. Then $i' \in \varphi(L)$ and there exists $i \in L$, $x \in M$ such that $\varphi(i) = i', \varphi(x) = x'$. Since L is an ideal in M, we get $x\alpha i \in L$ for every $i \in L$, $\alpha \in \Gamma$, $x \in M$. Now $x'\ \alpha\ i' = \varphi(x)\alpha\varphi(i) = \varphi(x\alpha i) \in \varphi(L)$(since $x\alpha i \in L$). Thus $\varphi(L)$ is a left Γ-hyperideal of $\varphi(M)$. Let $x', y' \in \varphi(M), \alpha \in \Gamma, i' \in \varphi(L)$. Then there exists $x, y \in M$, $i \in L$ such that $\varphi(x) = x', \varphi(y) = y', \varphi(i) = i'$. Since L is an Γ- hyperideal in M, we get $(x+i)\,\alpha y - x\alpha y \in L$ for every $x, y \in L, \alpha \in \Gamma, i \in L$. Consider $(x' + i')\ \alpha\ y' - x'\alpha\ y' = (\varphi(x) + \varphi(i))\ \alpha\ \varphi(y) - \varphi(x)\alpha\varphi(y) = \varphi(x + i)\ \alpha\ \varphi(y) - \varphi(x\alpha y) = \varphi(\,(x + i)\ \alpha y) - \varphi(x\alpha y) = \varphi(\,(x + i)\ \alpha\ y - x\alpha y) \in \varphi(L)$. Thus $\varphi(L)$ is a right Γ- hyperideal of $\varphi(M)$. Hence $\varphi(L)$ is an Γ- hyperideal of $\varphi(M)$.

Theorem 3.12. Let M and M′ be two Γ- hypernear-algebras over a field X. Let $\varphi : M \rightarrow M'$ be a Γ-hypernear-algebra homomorphism. If U′ is an Γ- hyperideal of M’, then $\varphi^{-}1$ (U′) is an Γ- hyperideal of M.

Proof. Put $U = \varphi^{-}1(U') = \{x \in M : \varphi(x) \in U'\}$. (1) Let $x, y \in U$ and $a,b \in X$. Then $\varphi(x),\varphi(y) \in U'$. This implies that $a\varphi(x)$, $b\varphi(y) \in U'$. Now U′ is an Γ-hyperideal, then $a\varphi(x) + b\varphi(y) \in U' \Rightarrow \varphi(ax) + \varphi(by) \in U' \Rightarrow \varphi(ax + by) \in U' \Rightarrow ax + by \in \varphi^{-}1(U') = U$. Therefore for every $x, y \in \varphi^{-}1(U'), a,b \in F$, we get $ax + by \in \varphi^{-}1(U')$. Thus $\varphi^{-}1(U')$ is a linear subspace of M. (2) Let $x \in M, i \in U, \alpha \in \Gamma$. Then $\varphi(i) \in U', \varphi(x) \in M'$. Now U′ is an Γ-hyperideal in M’, then $\varphi(x)\alpha\varphi(i) \in U' \Rightarrow \varphi(x\alpha i) \in U' \Rightarrow x\alpha i \in U$. Therefore for every $x \in M, i \in U$, we get $x\alpha i \in U$. Thus $\varphi^{-}1(U')$ is a left Γ- hyperideal of M. (3) Let $x, y \in M$, $i \in U$, $\alpha \in \Gamma$. Then $\varphi(i) \in U', \varphi(x),\varphi(y) \in M', \alpha \in \Gamma \Rightarrow (\varphi(x) + \varphi(i))\ \alpha\ \varphi(y) - \varphi(x)\alpha\varphi(y) \in U' \Rightarrow \varphi(x + i)\ \alpha\ \varphi(y) - \varphi(x\alpha y) \in U' \Rightarrow \varphi(\,(x + i)\ \alpha\ y - x\alpha y) \in U' \Rightarrow (x + i)\ \alpha\ y - x\alpha y \in U$. Thus $\varphi^{-}1(U')$ is a right Γ- hyperideal of M. Hence $\varphi^{-}1(U')$ is an Γ- hyperideal of M.

Definition 3.13. Let I be an Γ-hyperideal of a Γ- hypernear-algebra M over a field X. Then M/I is defined by $M/I = \{x + I : x \in M\}$, and M/I is called the quotient set.

Theorem 3.14. Let I be an Γ-hyperideal of a Γ- hypernear-algebra M over a ûeld X. Then the set M/I is a Γ -hypernear-algebra over X with respect to the operations defined by

$(x + I) + (y + I) = (x + y) + I$, $(x + I) = \lambda x + I$, $(x + I)\alpha(y + I) = x\alpha y + I$ for every $x, y \in M$, $\lambda \in X$, $\alpha \in \Gamma$.

Proof. Clearly first two operations are well defined. Now we prove that the third operation is well defined. Suppose that $x + I = x' + I$, $y + I = y' + I$, where $x + I, x' + I, y + I, y' + I \in M/I$. Then $x - x', y - y' \in I$. Put $x - x' = i_1$, $y - y' = i_2$, where $i_1,i_2 \in I$. So $x = x' + i_1$, $y = y' + i_2$. This implies that $x\alpha y = (x' + i_1)\alpha\,(y' + i_2) = (x' + i_1)\alpha\ y' + (x' + i_1)\alpha\ i_2$ which implies $x\alpha y - x'\alpha y' = (x' + i_1)\alpha\ y' + (x' + i_1)\alpha$

$i_2 - x'\alpha y = (x' + i_1)\alpha\, y' - x'\alpha y' + x' + i_1)\alpha\, i_2 \in I$. Therefore $x\alpha y - x'\alpha y' \in I$. This implies that $x\alpha y + I = x'\alpha y' + I$, and so $(x + I)\alpha(y + I) = (x' + I)\alpha(y' + I)$.

Thus the third operation is well defined. We shall show that M/I is a Γ-hypernear-algebra. A direct computation shows that M/I is a linear space over the field X. Let x + I, y + I, z + I ∈ M/I, α, β ∈ Γ, λ ∈ X where x, y,z ∈ M. Then:

$((x + I)\alpha(y + I))\beta(z + I) = (x\alpha y + I)\, \beta\, (z + I) = (x\alpha y)\, \beta\, z + I = x\alpha(y\, \beta\, z) + I = (x + I)\alpha(y\, \beta\, z + I) = (x + I)\alpha((y + I)\, \beta\, (z + I))$. Now $(x + I)\, \alpha\, ((y + I) + (z + I)) = (x + I)\, \alpha((y+z) + I) = x\, \alpha\, (y+ z) + I = (x\alpha y + x\alpha z) + I = (x\alpha y + I) + (x\alpha z + I) = (x + I)\alpha(y + I) + (x + I)\alpha(z + I)$, $(\lambda(x + I))\alpha(y + I) = (\lambda x + I)\alpha(y + I) = (\lambda x)\alpha y + I = \lambda(x\alpha y) + I = \lambda(x\alpha y + I) = \lambda((x + I)\alpha(y + I))$. Hence M/I is a Γ -hyper near-algebra over the field X.

This Γ-hypernear-algebra is called a Quotient Γ- hypernear-algebra.

Theorem 3.15. Let I be an ideal of a Γ- hypernear-algebra M over a field X, then the quotient Γ-hypernear-algebra M/I is the homomorphic image of M.

Proof. Let $\varphi : M \to M/I$ be a mapping defined by $\varphi(x) = x + I$ for every $x \in M$. Let $x, y \in M, \lambda \in F, \alpha \in \Gamma$. Suppose that $x = y$. Then $x + I = y + I$ implies that $\varphi(x) = \varphi(y)$. Therefore φ is well defined. Now $\varphi(x + y) = (x + y) + I = (x + I) + (y + I) = \varphi(x) + \varphi(y)$, $\varphi(x\alpha y) = x\alpha y + I = (x + I)\alpha(y + I) = \varphi(x)\alpha\varphi(y)$, $\varphi(\lambda x) = \lambda x + I = \lambda(x + I) = \lambda\varphi(x)$. Therefore φ is a Γ–hyper near-algebra homomorphism. Let $x + I \in M/I$. Then $x \in M$. For this $x \in M$ we have $\varphi(x) = x + I$. Thus for each $x + I \in M/I$ there exists $x \in M$ such that $\varphi(x)=x+I$. Therefore φ is onto. Hence M/I is a homomorphic image of M.

Theorem 3.16. Let M and M′ be two Γ- hyper near-algebras over a field F. Let $\varphi : M \to M'$ be a Γ-hypernear-algebra homomorphism with kernel I. Then φ(M) is isomorphic to M/I.

Proof. The proof is straightforward

4.Fuzzy dot Γ- hypersub-near- algebras:

Definition 4.1 :

Let (F,X) be a fuzzy field of the field X, M be a Γ- hypernear- algebra over X and μ be a fuzzy set of M. Then (μ, M) is called fuzzy dot Γ-hypersub-near- algebra of M over a fuzzy field (F,X) if

(i) $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$

(ii) $\mu(\lambda x) \geq \min\{F(\lambda), \mu(x\}\}$

(iii) $\mu(x \gamma y) \geq \mu(x)\bullet \mu(y)$

(iv) $\mu((\lambda x)\, \alpha\, y) = \mu(\lambda\, (x\, \alpha\, y)) \geq \min\{F(\lambda), \mu(x), \mu(y)\}$ for all $x, y \in M$, $\lambda \in X$ and $\gamma,\alpha \in \Gamma$.

Theorem 4.2 :

Let (μ, M) and (σ, M) be two fuzzy dot Γ-hyper sub-near-algebras of a Γ-hypernear-algebra M over a fuzzy field (F,X). Then (μ, M) ∩ (σ, M) is also a fuzzy dot Γ-hypersub-near- algebra over a fuzzy field (F, X)

Proof.

Let x, y ϵ M, λ ϵ X and γ, α ϵ Γ.

(i) $(\mu\cap\sigma)(x-y) = \min\{\mu(x-y), \sigma(x-y)\}$
$\geq \min\{\min\{\mu(x), \mu(y)\}, \min\{\sigma(x), \sigma(y)\}\}$
$= \min\{\min\{\mu(x), \sigma(x)\}, \min\{\mu(y), \sigma(y)\}$
$= \min\{(\mu\cap\sigma)(x), (\mu\cap\sigma)\{y)\}$

(ii) $(\mu\cap\sigma)\lambda(x) = \min\{\mu(\lambda x), \sigma(\lambda x)\}$
$\geq \min\{\min\{F(\lambda), \mu(x)\}, \min\{F(\lambda), \sigma(x)\}\}$
$= \min\{F(\lambda), \min\{\mu(x), \sigma(x)\}$
$= \min\{F(\lambda), \mu\cap\sigma(x)\}$

(iii) $(\mu\cap\sigma)(x\gamma y) = \min\{\mu(x\gamma y), \sigma(x\gamma y)\}$
$\geq \min\{\mu(x)\bullet\mu(y), \sigma(x)\bullet\sigma(y)\}$
$= \min\{\mu(x), \sigma(x)\}\bullet\min\{\mu(y), \sigma(y)\}$
$= (\mu\cap\sigma)(x)\bullet(\mu\cap\sigma)(y)$

(iv) $(\mu\cap\sigma)((\lambda x)\alpha y) = \min\{\mu((\lambda x)\alpha y), \sigma((\lambda x)\alpha y)\}$
$\geq \min\{\min\{F(\lambda), \mu(x), \mu(y)\}, \min\{F(\lambda), \sigma(x), \sigma(y)\}\}$ $= \min\{F(\lambda), \min\{\mu(x), \sigma(x)\}, \min\{\mu(y), \sigma(y)\}$
$= \min\{F(\lambda), (\mu\cap\sigma)(x), (\mu\cap\sigma)\{y)\}$.

Hence $(\mu\cap\sigma)$ is a fuzzy dot Γ-hypersub-near-algebra of M.

Theorem 4.3:

Let M be a Γ-hypernear- algebra over a field X. Let (μ, M) be a fuzzy dot Γ-hypersub-near- algebra of M over the fuzzy field (F, X). Let A be a subset of M. Then A is a sub Γ-hypernear-algebra of M if and only if χ_A is a fuzzy dot Γ-hypersub-near- algebra of M over a fuzzy field (F, X).

Proof.

Let A be a hypersub-algebra of M.

Case (i): Let x, y ϵ A. Then x – y, λ x ϵ A where λ ϵ X. Therefore $\chi_A(x-y) = 1$;

$\chi_A(\lambda x) = 1.$
$\chi_A(x-y) = 1 \geq \min\{\chi_A(x), \chi_A y)\}.$
$\chi_A(x\gamma y) = 1 \geq \chi_A(x)\bullet\chi_A(y)$
$(\lambda x)\alpha y = \lambda(x\alpha y)$ implies $\chi_A((\lambda x)\alpha y) = \chi_A(\lambda(x\alpha y)) \geq \min\{F(\lambda), \chi_A(x), \chi_A(y)\}.$

Case (ii) If x ϵ A, y ∉ A. Then x –y ∉ A.

Therefore, $\chi_A(x-y) = 0 = \min\{\chi_A(x), \chi_A(y)\}.$
$\chi_A(\lambda x) = 1 \geq \min\{F(\lambda), \chi_A(x)\}.$
$\chi_A(x\gamma y) = 0 = \chi_A(x)\bullet\chi_A(y)$ $\chi_{A(}(\lambda x)\alpha y) = 0$
$= \min\{F(\lambda), \chi_A(x), \chi_A(y)\}.$

In a similar manner one can verify the result for other cases x ∉A, y Є A and x ∉ A, y ∉ A. Thus χ_A is a fuzzy dot Γ-hypersub-near- algebra of M

Conversely assume that χ_A is a fuzzy dot Γ-hypersub-near- algebra of M. We claim that A is a sub Γ-hypernear algebra of M.

(i) (a) If x, y ϵ A then $\chi_A(x-y) \geq \min\{\chi_A(x), \chi_A(y)\} = \min\{1, 1\} = 1$. This implies that x – y ϵ A.

(b) If x ϵ A, y ∉ A then $\chi_A(x-y) \geq \min\{\chi_A(x), \chi_A(y)\} = \min\{1,0\} = 0$. This implies that x – y ∉ A

(c) If x ∉ A, y ϵ A then $\chi_A(x-y) \geq \min\{\chi_A(x), \chi_A(y)\} = \min\{0,1\} = 0$. This implies that x – y ∉ A

(d) If x ∉ A, y ∉ A then $\chi_A(x-y) \geq$ min

$\{\chi_A(x), \chi_A(y)\} = \min\{0,0\} = 0$. This implies that $x - y \notin A$

(ii) If $x \in A$, $\chi_A(\lambda x) \geq \min\{F(\lambda), \chi_A(x)\} = \min\{F(\lambda), 1\} = 1$.

This implies that $\lambda x \in A$.

Thus A is a sub Γ-hypernear-algebra of M.

Proposition 4.4 :

A fuzzy set μ of M is a fuzzy dot Γ-hypersub-near-algebra of M if and only if the non-empty level subset $U(\mu,t)=(x \in M/\mu(x) \geq t\}$ is a sub Γ-hypernear-algebra of M for all $t \in [0,1]$.

Proposition 4.5 :

If μ is a fuzzy dot Γ-hypersub-near-algebra of M then $U(\mu ; 1)= \{(x \in M / \mu(x)=1\}$ is either empty or is a sub Γ-hypernear-algebra of M.

Theorem 4.6 :

Let M and M' be Γ-hyper near-algebras over a field X. Let $f: M \rightarrow M'$ be an onto homomorphism. If μ is a fuzzy dot Γ-hypersub-near- algebra of M then the image f[μ] of v under f is fuzzy dot Γ-hypersub-near-algebra of M'.

Proof.

We have $f[\mu](y') = \sup_{x \in f^{-1}(y')} \mu(x)$.

Let $x', y' \in M'$. Let $x_o \in f^{-1}(x')$ and $y_o \in f^{-1}(y')$ be such that $\mu(x_o)= \sup_{z \in f^{-1}(x')} \mu(z)$ and $\mu(y_o) = \sup_{z \in f^{-1}(y')} \mu(z)$.

(i) $f[\mu](x' - y') = \sup_{z-y \in f^{-1}(x' - y')} \mu(z - y)$

$= \sup_{x-y \in f^{-1}(x') - f^{-1}(y')} \mu(z - y)$

$= \sup_{z \in f^{-1}(x') - f^{-1}(y')} \mu(z)$

$\geq \mu(x_o - y_o)$

$\geq \min\{\mu(x_o), \mu(y_o)\}$

$= \min\{\sup_{z \in f^{-1}(x')} \mu(z), \sup_{z \in f^{-1}(y')} \mu(z)\}$

$= \min\{f[\mu](x'), f[\mu](y')\}$

(ii) $f[\mu](\lambda x') = \sup_{z \in f^{-1}(\lambda x')} \mu(z) \geq \mu(\lambda x_o)$

$\geq \min\{F(\lambda), \mu(x_o)\}$

$= \min\{F(\lambda), \sup_{z \in f^{-1}(x')} \mu(z)\}$

. $= \min\{F(\lambda), f[\mu](x')\}$

(iii) $f[\mu](x' \gamma y') = \sup_{z \in f^{-1}(x' \gamma y')} \mu(z)$

$= \sup_{z \in f^{-1}(x') \gamma f^{-1}(y')} \mu(z)$

$\geq \mu(x_o \gamma y_o)$

$\geq \mu(x_o) \bullet \mu(y_o)$

$= \sup_{z \in f^{-1}(x')} \mu(z) \bullet \sup_{z \in f^{-1}(y')} \mu(z)$

. $= f[\mu](x') \bullet f[\mu](y')$

(iv) $f[\mu]((\lambda x') \alpha y') \geq \min\{F(\lambda), f[\mu](x'), f[\mu](y')\}$.

Thus f[μ] is fuzzy dot Γ-sub-near- algebra of M' .

Theorem 4.7 :

Let $g: M \rightarrow M'$ be an onto homomorphism. If ν is a fuzzy dot Γ-hypersub-near-algebra of M' then the pre-image $g^{-1}[\nu]$ defined by $g^{-1}[\nu](x)=\mu(x)= \nu(g(x))$ for all $x \in M$ is also a fuzzy dot Γ-hypersub-near- algebra of M.

Proof.

Let $x, y \in M$; $\lambda \in X$; $\gamma, \alpha \in \Gamma$.

(i) $g^{-1}[\nu](x - y) = \nu(g(x - y)) = \nu(g(x) - g(y))$
$\geq \min\{\nu(g(x), \nu(g(y)\}$
$= \min\{g^{-1}[\nu](x), g^{-1}[\nu](y)\}$.

(ii) $g^{-1}[\nu](\lambda x) = \nu(g(\lambda x)) = \nu(\lambda(g(x))$
$\geq \min\{F(\lambda), \nu(g(x))\}$
$= \min\{F(\lambda), g^{-1}[\nu](x)\}$.

(iii) $g^{-1}[\nu](x \alpha y) = \nu(g(x \alpha y))$
$= \nu(g(x) \alpha g(y))$
$\geq \nu(g(x) \bullet \nu(g(y)$
$= g^{-1}[\nu](x) \bullet g^{-1}[\nu](y)$.

(iv) $g^{-1}[\nu]((\lambda x) \alpha y) = \nu(g((\lambda x) \alpha y))$
$= \nu(g(\lambda x) \alpha g(y))$
$= \nu(\lambda g(x) \alpha g(y))$
$= \nu(\lambda(g(x) \alpha g(y)))$
$\geq \min\{F(\lambda), \nu(g(x)), \nu(g(y)\}$
$= \min\{F(\lambda), g^{-1}[\nu](x), g^{-1}[\nu](y)\}$

Thus $g^{-1}[\nu]$ is fuzzy dot Γ-hypersub-near-algebra of M.

Theorem 4.8 :

Let M be Γ-hypersub-near- algebra over field X. Let I be an ideal of M. If μ is a fuzzy dot Γ-hypersub-near-algebra of M then the fuzzy set μ of M/I defined by $\mu'(m + I) = \sup_{x \in I} \mu(m + x)$ is fuzzy dot Γ-hypersub-near- algebra of M/I.

Proof.

Let $m_1, m_2 \in M$ be such that $m_1 + I = m_2 + I$. Then $m_2 - m_1 \in I$. So $m_2 - m_1 = m$ for some $m \in I$.

Now $\mu'(m_2 + I) = \sup_{m' \in I} \mu(m_2 + m') = \sup_{m' \in I} \mu(m_1 + m' + m) = \sup_{m + m' = t \in I} \mu(m_1 + t) = \mu'(m_1 + I)$. Therefore, μ' is well defined.

Let $m_1 + I, m_2 + I \in M/I$.

(i) $\mu'((m_2 + I) - (m_1 + I)) = \mu'(m_2 - m_1 + I))$
$= \sup_{u - v \in I} \mu(m_2 - m_1 + u - v)$
$= \sup_{u, v \in I} \mu((m_2 + u) - (m_1 + v))$
$\geq \sup_{u, v \in I} \min\{\mu(m_2 + u), \mu(m_1 + v)\}$
$= \min\{\sup_{u \in I} \mu(m_2 + u), \sup_{v \in I} \mu(m_1 + v)\} = \min\{\mu'(m_2 + I), \mu'(m_1 + I))\}$

(ii) $\mu'(\lambda(m + I) = \mu'(\lambda m + I)$
$= \sup_{t \in I} \mu(\lambda m + t)$
$= \sup_{t \in I} \min\{F(\lambda), \mu(m + t)\}$
$= \min\{F(\lambda), \sup_{t \in I} \mu(m + t)\}$
$= \min\{F(\lambda), \sup \mu'(m + t)\}$

(iii) Let $m_1 + I, m_2 + I \in M/I$ and $\gamma \in \Gamma$.
$\mu'((m_1 + I) \gamma (m_2 + I)) = \mu'(m_1 \gamma m_2 + I))$
$= \sup_{t \in I} \mu(m_1 \gamma m_2 + t)$
$= \sup_{t \in I} \mu((m_1 + t) \gamma (m_2 + t))$
$\geq \sup_{t \in I} \mu(m_1 + t) \bullet \mu(m_2 + t)$
$= \sup_{t \in I} \mu(m_1 + t) \bullet \sup_{t \in I} \mu(m_2 + t)$
$= \mu'(m_1 + I) \bullet \mu'(m_2 + I)$.

(iv) Let $m_1 + I, m_2 + I \in M/I$, $\gamma \in \Gamma$, and $\lambda \in X$.
$\mu'(\lambda(m_1 + I) \gamma (m_2 + I))$
$= \mu'(\lambda((m_1 + I) \gamma (m_2 + I)))$
$= \mu'(\lambda(m_1 \gamma m_2 + I))$
$= \sup_{t \in I} \mu(\lambda(m_1 + t) \gamma (m_2 + t)$
$\geq \sup_{t \in I} \min\{F(\lambda), \mu(m_1 + t), \mu(m_2 + t)\}$
$= \min\{F(\lambda), \sup_{t \in I} \mu(m_1 + t), \sup_{t \in I}$

$\mu(m_2 + t)\}$
$= \min\{F(\lambda), \mu'(m_1 + I), \mu'(m_2 + I))\}$.

Thus μ' is fuzzy dot Γ-hypersub-near- algebra of M/I.

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