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Fixed Point Theorems of T-Kannan Contractive on T-Stability of Picard Iteration in Cone Metric Spaces

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Abstract

The purpose of this paper is to obtain sufficient conditions for the existence of a unique fixed point of T-Kannan type mapping on complete cone metric spaces with the T – stability of Picard's iteration procedures.

Key words: Fixed point, T-Kannan contractive mappings, complete cone metric space, sequentially convergent, Picard's iteration.

1. Introduction

The purpose of this paper is to analyze the existence (and uniqueness) of fixed points of T - Kannan type contractive mappings S defined on a complete cone metric space (M, d), as well as, T - Chaterjea mappings which are introduced here. Our results generalize the respective theorems given[1,3].

2. Definitions and preliminary results:

In this section we call the definition of cone metric space and some of their properties[1]. The following notions will be used in order to prove the main results.

Definition 2.1. Let E be a real Banach space and P a subset of E. The set P is called a cone if and only if:

P1- P is closed, non empty and $P\neq\{0\}$;
P2- $a, b\in R$, $a, b\geq 0$, $x, y \in P \Rightarrow ax+by \in P$;
P3- $x\in P$ and $-x\in P \Rightarrow x=0$.

Given a cone $P\subset E$, we define a partial ordering $\leq$ with respect to P by $x\leq y$ if and only if $y-x\in P$. We write $x< y$ to indicate that $x \leq y$ but $x \neq y$, while $x\ll y$ if and only if $y-x \in \text{Int } P$, where Int P denotes the interior of the set P.

Definition 2.2. Let E be a Banach space and $P \subset E$ a cone. The cone P is called normal if there is a number $K > 0$ such that for all x, $y \in E$, $0 \le x \le y$ implies $\|x\| \le K\|y\|$.

The least positive number K satisfying the above Inequality is called the normal constant of P.

In the following we always suppose that E is a Banach space , P is a cone in

E with Int $P \neq \emptyset$ and $\le$ is partial order in g with respect to P.

Definition 2.3. Let M be a non empty set. Suppose that the mapping d: $M \times M \to E$ satisfies:

d1- $0 < d(x,y)$ for all x , $y \in M$ and $d(x,y) = 0$ if and only if x=y;

d2- $d(x,y) = d(y, x)$ for all x, $y \in M$;

d3- $d(x,y) \le d(x,z) + d(y, z)$ for all x, y, $z \in M$.

Then, d is called a cone metric on M and (M, d) is called a cone metric space.

Notice that the notion of cone metric space is more general than the corresponding of metric space. Examples of cone metric spaces can be found[1,4].

Definition 2.4. Let (M, d) be a cone metric space . Let (x_n) be sequence in M and $x \in M$.

(i) (x_n) converges to x iff or every $c \in E$ with $0 \ll c$, there is an n_0 such that for all $n \ge n_0$, $d(x_n,x) \ll c$. We denote this by

$$\lim_{n\to\infty} x_n = x \text{ or } x_n \to x, (n\to\infty).$$

(ii) If for any $c \in E$ with $0 \ll c$, there is an n_0 such that

For all n, $m \ge n_0$, $d(x_n,x_m) \ll c$, then (x_n) is called a Cauchy sequence in M.

(M, d) is called a complete cone metric space, if every Cauchy sequence in M is convergent in M.

The following result will be useful for us to prove our main results.

Lemma 2.5.[1] Let (M, d) be a cone metric space, $P \subset E$ a normal cone with normal constant K. Let (x_n) , (y_n) be sequences in M and x , $y \in M$.

(i) (x_n) converges to x if and only if $\lim_{n\to\infty} d(x_n, x) = 0$

(ii) If (x_n) converges to x and (x_n) converges to y then x = y. That is the limit of (x_n) is unique;

(iii) If (x_n) converges to x , then (x_n) is Cauchy sequence;

(iv) (x_n) is a Cauchy sequence if and only if

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0;$$

(v) If $x_n \to x$ and $y_n \to y$, $(n\to\infty)$ then

$$d(x_n,y_n) \to d(x, y).$$

Definition 2.6. Let (M, d) be a cone metric space, P a normal cone with normal constant K and $T : M \to M$. Then

(i) T is said to be continuous if $\lim_{n\to\infty} x_n = x$

implies that $\lim_{n\to\infty} T(x_n)=T(x)$, for all (x_n) in M;

(ii) T is said to be subsequentially convergent, if we have for every sequence (y_n) that $T(y_n)$ is convergent , implies (y_n) has a convergent subsequence;

(iii) T is said to be sequentially convergent if we have, for every sequence (y_n), If $T(y_n)$ is convergent , then (y_n) also is convergent.

Definition 2.7. Let (M, d) be a cone metric space and T, S: M $\to$M two functions. S is said to be T-contraction if there is b, c$\in$[0,1/2) constant such

$$d(TSx, TSy)\leq \max\{b[d(Tx, TSx)+d(Ty, TSy)], c[d(Tx, TSy)+d(Ty, TSx)]$$

Main Result

Theorem 3.1. Let (M, d) be a complete cone metric space, P be a normal cone with normal constant K, in addition let T : M $\to$M be a one to one, continuous function and S:M$\to$M a T–Contraction. Then,

(1) For every $x_0\in M$

$$\lim_{n\to\infty} d(TS^n x_0 ,\ \ TS^{n+1}x_0) = 0;$$

(2) There is v $\in$ M such that

$$\lim_{n\to\infty} TS^n x_0 = v$$

(3) If T is subsequentially convergent, then $(S^n x_0)$ has a convergent subsequence;

(4) There is a unique u $\in$ M such that

Su=u;

(5) If T is sequentially convergent, then for each $x_0 \in M$ the iterate sequence $(S^n x_0)$ converges to u.

Proof . Let x_0 be an arbitrary point in M. We define the iterate sequence (x_n) By $x_{n+1}= Sx_n=S^n x_0$.We have

$$d(TSx_n,TSx_{n+1})\leq b[d (TSx_n, TSx_n) +d(Tx_{n+1}, TSx_{n+1})]$$

$$\leq b[d (TSx_{n-1} , TSx_n)+d (TSx_n, TSx_{n+1})$$

$$d(TSx_n,TSx_{n+1})\leq \frac{b}{1-b} d\ (TSx_{n-1} , TSx_n)$$

$$=(\frac{b}{1-b}) d (TSx_{n-1} , TSx_n)$$

$$d(TSx_n,TSx_{n+1})\leq (\frac{b}{1-b})^n d (TSx_0, TSx_1)$$

$$\|d(TSx_n,TSx_{n+1})\|\leq (\frac{b}{1-b})^n k\|(TSx_0, TSx_1)\|$$

Where k is normal constant of M. hence

$$\lim_{n\to\infty} d\|TSx_n,TSx_{n+1}\| =0$$

$$\lim_{n\to\infty} d\ (TS^n x_0, TS^{n+1} x_0) =0$$

By inequality (3.1), for every m,n $\in$ N with m > n we have

$$d(TSx_n, TSx_m)\leq d(TSx_n, TSx_{n+1})+ \ldots +d (TSx_{m-1}, TSx_m)$$

$$\leq [\left(\frac{b}{1-b}\right)^n + \cdots + \left(\frac{b}{1-b}\right)^{m-1}] d(TSx_0 , TSx_1)$$

$$d(TSx_n, TSx_m)= \left(\frac{b}{1-b}\right)^n\ 1 - (\frac{1}{1-b}) d (TSx_0,TSx_1)$$

$$\|\ d(TSx_n, TSx_m \| \leq (\frac{b}{1-b})^n \frac{k}{1-\frac{b}{1-b}} \|d (TSx_0, TSx_1)\|$$

$$\lim_{n,m\to\infty} d(TSx_n, TSx_m) =0$$

Hence $(TS^n x_0)$ is a Cauchy sequence in M and since M is complete cone metric, there is V$\in$M such that

$$\lim_{n\to\infty} TS^n (x_0) = v$$

The rest of the proof is similar to the proof of theorem 3.3 [1]

Corollary 3.2[1] (Theorem 3). Let (M, d) be a complete cone metric space, P a normal cone with normal constant K. Suppose that the mapping $S : M \to M$ satisfies the contractive condition

$$d(Sx, Sy) \leq b[d(x, Sx)+d(y, Sy)]$$

for all x, $y \in M$ and $b \in [0,1/2)$. Then S has a unique fixed point in M and for any $x_0 \in M$, iterative sequence $(S^n x_0)$ converges to the fixed point.

If we take M =R in the Theorem 3.3 [1] we obtain the following:

Corollary 3.3[3] (Theorem 2.1). Let (M, d) be a complete metric space and T , S : $M \to M$ be mappings such that T is continuous ,one to one and subsequentially convergent. If $b \in [0, 1/2)$ and

$$d(TSx, TSy) \leq b[d(Tx, TSx)+d(Ty, TSy)]$$

for all x , $y \in M$. Then S has a unique point and if T is sequentially convergent, then for every $x_0 \in M$ the sequence iterates$(S^n x_0)$ converges to the fixed point of S.

If we take M= R and Tx = x in the Theorem 3.3[1], we obtain the following result given by Kannan[2].

Corollary 3.4. Let (M,d) be a complete metric space and S :$M \to M$ a mapping. If there exists $b \in [0,1/2)$ such that

$$d(Sx, Sy) \leq b[d(x, Sx)+d(y, Sy)]$$

for all x, $y \in M$. Then S has a unique fixed point and $(S^n x_0)$ converges to the fixed point of S for all $x_0 \in M$.

The following result extend the Theorem 4 [1].

Theorem 3.5. Let (M, d) be a complete cone metric space, P be a normal cone with normal constant K , let in addition T: $M \to M$ be a continuous, one to one function and $S: M \to M$ a T–Contraction.Then

(1) For every $x_0 \in M$

$$\lim_{n\to\infty} d(TS^n x_0, TS^{n+1} x_0) = 0$$

(2) There is $v \in M$ such that

$$\lim_{n\to\infty} TS^n x_0 = v$$

(3) If T is subsequentially convergent, then $(S^n x_0)$ has a convergent subsequence.

(4) There is a unique $u \in M$ such that

$$Su=u.$$

(5) If T is sequentially convergent, then for each $x_0 \in M$, $(S^n(x_0))$ converges to u.

Proof. Let x_0 be an arbitrary point in M. We define the iterative sequence (x_n) By $x_{n+1}=Sx_n=S^n x_0$. Since S is a T–Contraction, we have

$$d(Tx_n,Tx_{n+1})=d(TSx_{n-1}, TSx_n)$$
$$\leq c[d(Tx_{n-1}, TSx_n)+d(Tx_n, TSx_{n-1})]$$
$$d(Tx_n,Tx_{n+1}) \leq \frac{c}{1-c} d(Tx_{n-1}, Tx_n)$$

$$d(TS^n{}_{x0},TS^{n+1}{}_{x0}) \le \left(\frac{c}{1-c}\right)^n d(Tx_0, TSx_0)$$

$$\|d(TS^n{}_{x0},TS^{n+1}{}_{x0})\| \le \left(\frac{c}{1-c}\right)^n k\| d(Tx_0, TSx_0)\|$$

$$\lim_{n\to\infty}\|d(TS^n{}_{x0}, TS^{n+1}x_0)\| = 0$$

Hence

$$\lim_{n\to\infty} d(TS^n{}_{x0}, TS^{n+1}x_0) = 0$$

By inequality (3.8) for every n, m∈N with n>m we have,

$$d(Tx_m, Tx_n) \le d(Tx_m, Tx_{m+1}) + \ldots + d(Tx_{n-1}, Tx_n)]$$

$$\le [(\tfrac{c}{1-c})^m + \ldots + [(\tfrac{c}{1-c})^{n-1}]\, d(Tx_0, TSx_0)$$

$$d(TS^m{}_{x0}, TS^n{}_{x0}) \le [(\tfrac{c}{1-c})^m [\frac{1}{(\frac{c}{1-c})}]\, d(Tx_0, TSx_0)$$

$$\|d(TS^n{}_{x0},TS^m{}_{x0})\| \le (\tfrac{c}{1-c})^m \frac{k}{1-\frac{c}{1-c}}\|d(Tx_0, TSx_0)\|$$

$$\lim_{n,m\to\infty}\|d(TS^n x_0, TS^m{}_{x0})\| = 0$$

Now, if T is sub-sequentially convergent, $(S^n x_0)$ has a convergent subsequence $\{n_i\}$ such that $\lim_{i\to\infty} S^{ni}x_0=u$. Since T is continuous, therefore

$$\Rightarrow T(\lim_{i\to\infty} S^{ni}x_0) = Tu.$$

$$\Rightarrow \lim_{i\to\infty} T\,S^{ni}x_0 = Tu$$

Now

$$d(TSu, Tu) = \lim_{i\to\infty} d(T\,S(S^{ni}x_0), TS^{ni}x_0)$$

$$\le \lim_{i\to\infty} b[d(TS^{ni}x_0, TS^{ni}x_0) + d(TS^{n-1}(x_0), TS^{ni}x_0)$$

$$\le b[d(Tu, TSu) + \lim_{i\to\infty} d(TS^{ni-1}x_0, TS^{ni}x_0)$$

$$d(TSu, Tu) \le \frac{b}{1-b}\lim_{i\to\infty} d(TS^{ni-1}{}_{x0}, TS^{ni}{}_{x0})$$

$\Rightarrow$ d(TSu, Tu) $\to$ 0 as i $\to\infty$.

$\Rightarrow$ TSu = Tu

So that Su = u.

Hence proved

References

1. Guang, H. L. and Xian, Z., Cone metric spaces and fixed point theorems of contractive mappings, *J. Math. Anal. Appl. 332*, 1468-1476 (2007).
2. Kannan, R., Construction of fixed points of a class of nonlinear mappings *J. Math. Anal. Appl. 41*, 430–438 (1973).
3. Mbarki, A. B., Fixed points for Near-contractive type multi-valued mappings, *Southwest Journal of Pure and App. Mathematics*, 18-25 (2003).
4. Rakotch, E., A note on contraction mappings, *Proc. Amer. Math. Soc., 13*, 459-462 (1962).