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A new type of Generalized Fixed Point Theorem in a Complete Metric Space

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Abstract

In the present paper, we prove a fixed point theorem in the form of rational expression. Our result generalizes and extends some well known previous results.

Key words: Fixed point, complete metric space, fixed point theorem.

Introduction

In 1922, the Polish mathematician Stefan Banach proved a theorem which ensures, under appropriate conditions, the existence and uniqueness of a fixed point. His result is called Banachs fixed point theorem. This result provides a technique for solving variety of applied problems in mathematical science and engineering. Many authors like Kannon[2], Reich[3], Chatterjee[4], Fisher[5] and Yadav[6] have extended, generalized and improved Banach fixed point theorem in different ways.

The aim of this paper is to obtain a fixed point theorem in the form of rational expression. we extended the work of and then show that the results of Banach, Kannon, Reich, Chatterjee and Fisher are special cases of our theorem.

Throughout this paper the complete metric space (X, d) is denoted by X.

Theorem: Let f be a continuous self mapping defined on a complete metric space X with

$$d(f_x, f_y) \le k_1 \frac{d(x, fx)d(y, fy) + d(x, fy)d(y, fx)}{d(x, fx) + d(y, fy) + d(x, fy) + d(y, fx)} +$$

$$k_2 \frac{d(x, fx)d(y, fx) + d(y, fy)d(x, fy)}{d(x, fx) + d(y, fx) + d(y, fy) + d(x, fy)} + k_3 \frac{d(x, fx)d(x, fy) + d(y, fx)d(y, fy)}{d(x, fx) + d(x, fy) + d(y, fx) + d(y, fy)} +$$

$k_4\,[d(x, fx) + d(y, fy)] + k_5\,[d(x, fy) + d(y, fx)] + k_6\,[d(x, fy) + d(y, f_y)] + k_7\, d(x, y)$.
$\forall$ x, y $\in$ X, x $\neq$ y and for $k_1, k_2, k_3, k_4, k_5, k_6, k_7 \in [0, 1)$ with $k_1 + k_2 + k_3 + 4k_4 + 4k_5 + 2k_6 + 2k_7 < 2$, then f has a unique fixed point.

Proof: Let $x_o \in X$ and define a sequence $\{x_n\}$ in X such that $f^n_{x_0} = x_n \;\forall\; n \in I^+$.

Thus we set $fx_n = x_{n+1}$.

Now $d(x_{n+1}, x_n) = d(fx_n, fx_{n-1}) \le k_1 \dfrac{d(x_n, f_{x_n})d(x_{n-1}, f_{x_{n-1}}) + d(x_n, f_{x_{n-1}})d(x_{n-1}, f_{x_n})}{d(x_n, f_{x_n}) + d(x_{n-1}, f_{x_{n-1}}) + d(x_n, f_{x_{n-1}}) + d(x_{n-1}, f_{x_n})}$

$+ k_2 \dfrac{d(x_n, f_{x_n})d(x_{n-1}, f_{x_n}) + d(x_{n-1}, f_{x_{n-1}})d(x_n, f_{x_{n-1}})}{d(x_n, f_{x_n}) + d(x_{n-1}, f_{x_n}) + d(x_{n-1}, f_{x_{n-1}}) + d(x_n, f_{x_{n-1}})}$

$+ k_3 \dfrac{d(x_n, f_{x_n})d(x_n, f_{x_{n-1}}) + d(x_{n-1}, f_{x_n})d(x_{n-1}, f_{x_{n-1}})}{d(x_n, f_{x_n}) + d(x_n, f_{x_{n-1}}) + d(x_{n-1}, f_{x_n}) + d(x_{n-1}, f_{x_{n-1}})} + k_4\,[d(x_n, fx_n) + d(x_{n-1}, fx_{n-1})]$

$+ k_5\,[d(x_n, fx_{n-1}) + d(x_{n-1}, fx_n)] + k_6\,[d(x_n, fx_{n-1}) + d(x_{n-1}, fx_{n-1})] + k_7\, d(x_n, x_{n-1})$.

Or $d(x_{n+1}, x_n) \le k_1 \dfrac{d(x_n, x_{n+1})d(x_{n-1}, x_n) + d(x_n, x_n)d(x_{n-1}, x_{n+1})}{d(x_n, x_{n+1}) + d(x_{n-1}, x_n) + d(x_n, x_n) + d(x_{n-1}, x_{n+1})}$

$+ k_2 \dfrac{d(x_n, x_{n+1})d(x_{n-1}, x_{n+1}) + d(x_{n-1}, x_n)d(x_n, x_n)}{d(x_n, x_{n+1}) + d(x_{n-1}, x_{n+1}) + d(x_{n-1}, x_n) + d(x_n, x_n)}$

$+ k_3 \dfrac{d(x_n, x_{n+1})d(x_n, x_n) + d(x_{n-1}, x_{n+1})d(x_{n-1}, x_n)}{d(x_n, x_{n+1}) + d(x_n, x_n) + d(x_{n-1}, x_{n+1}) + d(x_{n-1}, x_n)} + k_4\,[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] +$
$k_5\,[d(x_n, x_n) + d(x_{n-1}, x_{n+1})] + k_6\,[d(x_n, x_n) + d(x_{n-1}, x_n)] + k_7\, d(x_n, x_{n-1})$.

i.e. $d(x_{n+1}, x_n) \le k_1 \dfrac{d(x_n, x_{n+1})d(x_{n-1}, x_n)}{d(x_{n-1}, x_n) + d(x_{n-1}, x_n)} + k_2 \dfrac{d(x_n, x_{n+1})d(x_{n-1}, x_{n+1})}{d(x_{n-1}, x_{n+1}) + d(x_{n-1}, x_{n+1})}$

$+ k_3 \dfrac{d(x_{n-1}, x_{n+1})d(x_{n-1}, x_n)}{d(x_{n-1}, x_{n+1}) + d(x_{n-1}, x_{n+1})} + k_4\,[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] + k_5\, d(x_{n-1}, x_{n+1}) + k_6\, d(x_{n-1}, x_n)$
$+ k_7\, d(x_n, x_{n-1})$.

$$\le \frac{k_1}{2}\, d(x_n, x_{n+1}) + \frac{k_2}{2}\, d(x_n, x_{n+1}) + \frac{k_3}{2}\, d(x_n, x_{n+1}) + k_4\,[d(x_n, x_{n+1}) +$$

$d(x_{n-1}, x_n)] + k_5\, d(x_{n-1}, x_{n+1}) + k_6\, d(x_{n-1}, x_n) + k_7\, d(x_n, x_{n-1})$.

i.e. $d(x_{n+1}, x_n)\left[1 - \left(\frac{k_1}{2} + \frac{k_2}{2} + \frac{k_3}{2} + k_4 + k_5\right)\right] \le d(x_{n-1}, x_n)\,[k_4 + k_5 + k_6 + k_7]$.

i.e. $d(x_{n+1}, x_n) \le \dfrac{k_4 + k_5 + k_6 + k_7}{1 - \frac{k_1}{2} - \frac{k_2}{2} - \frac{k_3}{2} - k_4 - k_5}\, d(x_{n-1}, x_n)$.

Continuing this process we get $d(x_{n+1}, x_n) \le \left(\dfrac{k_4 + k_5 + k_6 + k_7}{1 - \frac{k_1}{2} - \frac{k_2}{2} - \frac{k_3}{2} - k_4 - k_5}\right)^n d(x_0, x_1)$

$\le k^n\, d(x_0, x_1)$.

Where $k = \dfrac{k_4 + k_5 + k_6 + k_7}{1 - \frac{k_1}{2} - \frac{k_2}{2} - \frac{k_3}{2} - k_4 - k_5} < 1$, Since $k_1 + k_2 + k_3 + 4k_4 + 4k_5 + 2k_6 + 2k_7 < 2$.

Now for $m > n$, $d(x_n, x_m) \le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})$--------$d(x_{m-1}, x_m)$.

$\le (k^n + k^{n+1}$-------$k^{m-1})\, d(x_0, x_1)$.

So $d(x_n, x_m) \le \dfrac{k^n}{1-k}\, d(x_0, x_1)$

i.e. $d(x_n, x_m) \to 0$ as $n \to \infty$.

Thus $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exist a point $x \in X$ such that $x_n \to x$ as $n \to \infty$.

Again continuity of f gives $f(x) = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} fx_n = \lim_{n\to} x_{n+1} = x$.

Thus $f(x) = x$ i. e. x is a fixed point of f in X.

Now we show that x is unique. For suppose y be other fixed point such that $f(y) = y$,

Then $d(x, y) = d(f_x, f_y) \le k_1 \dfrac{d(x, f_y)d(y, f_y) + d(x, fy)d(y, f_x)}{d(x, f_x) + d(y, f_y) + d(x, f_y) + d(y, f_x)} +$

$k_2 \dfrac{d(x, f_y)d(y, f_x) + d(y, f_y)d(x, f_y)}{d(x, f_y) + d(y, f_x) + d(y, f_y) + d(x, f_y)} + k_3 \dfrac{d(x, f_x)d(x, f_y) + d(y, f_x)d(y, f_y)}{d(x, f_x) + d(x, f_y) + d(y, f_x) + d(y, f_y)}$

$+ k_4\,[d(x, f_x) + d(y, f_y)] + k_5\,[d(x, f_y) + d(y, f_x)] + k_6\,[d(x, f_y) + d(y, f_y)] + k_7\, d(x, y)$.

$$\leq k_1 \frac{d(x,x)d(y,y)+d(x,y)d(y,x)}{d(x,x)+d(y,y)+d(x,y)+d(y,x)} + k_2 \frac{d(x,x)d(y,x)+d(y,y)d(x,y)}{d(x,x)+d(y,x)+d(y,y)+d(x,y)}$$

$$+ k_3 \frac{d(x,x)d(x,y)+d(y,x)d(y,y)}{d(x,x)+d(x,y)+d(y,x)+d(y,y)} + k_4[d(x,x)+d(y,y)] + k_5[d(x,y)+d(y,x)]$$

$$+ k_6[d(x,y)+d(y,y)] + k_7\, d(x,y).$$

Or $d(x,y) \leq \frac{k_1}{2} d(x,y) + 2k_5 d(x,y) + k_6 d(x,y) + k_7 d(x,y)$.

so $d(x,y) \leq [\frac{k_1}{2} + 2k_5 + k_6 + k_7]\, d(x,y)$.

Which is a contradiction, because $k_1 + k_2 + k_3 + 4k_4 + 4k_5 + 2k_6 + 2k_7 < 2$.

Thus $d(x, y) = 0$ i. e. $x = y$. Hence x is the unique fixed point of f.

Remark:

(i) If $k_1 = 2_2 = k_3 = k_4 = k_5 = k_6 = 0$ then the theorem reduce to Banach[1].

(ii) If $k_1 = k_2 = k_3 = k_5 = k_6 = k_7 = 0$ then the theorem reduce to Kkannan[2].

(iii) If $k_1 = k_2 = k_3 = k_6 = 0$ then the theorem reduce to Reich[3].

(iv) If $k_1 = k_2 = k_3 = k_5 = k_6 = 0$ then the theorem reduce to Chatterjee[4].

(v) If $k_1 = k_2 = k_3 = k_4 = k_6 = 0$ then the theorem reduce to Fisher[5].

(vi) If $k_6 = 0$ then the theorem reduce to Yadav[6].

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