Ultra Scientist Vol. 27(3)A, 189-204 (2015).

Prime Cordial Labeling of Generalized Prism Graph $Y_{m,n}$

U. M. PRAJAPATI[1] and S. J. GAJJAR[2]

[1]Department of Mathematics, St. Xavier's College, Ahmedabad - 380009 (India)
E-mail: udayan64@yahoo.com
[2]General Department, Government Polytechnic, Himmatnagar - 383001 (India)
E-mail: gjr.sachin@gmail.com

(Acceptance Date 16th October, 2015)

Abstract

In this paper the authors have proved that prism graph $Y_{n,2}$ is prime cordial except n = 1, 2 and 4 and the graph $Y_{n,4}$ also prime cordial for n ≥ 3. We have also proved that the generalized prism graph $Y_{3,n}$, $Y_{5,n}$, $Y_{6,n}$ and $Y_{2p,n}$ (for odd prime p) are prime cordial for $n > 1$ and $Y_{4,n}$ is also prime cordial for $n > 2$.

Key words : Graph Labeling, Prime cordial Labeling, Generalized Prism graph.

AMS subject classification number: 05C78

1. Introduction

We consider only simple, finite, undirected and non-trivial graph $G = (V,E)$ with the vertex set V and the edge set E. The number of elements of V, denoted as $|V|$ is called the order of the graph G while the number of elements denoted as $|E|$ is called the size of the graph G. $Y_{m,n}$ denotes the generalized prism graph. For various graph theoretic notations and terminology we follow Gross and Yellen[1] whereas for number theory we follow D. M. Burton[2]. We will give brief summary of definitions and other information which are useful for the present investigations.

Definition 1.1: If the vertices of the graph are assigned values subject to certain conditions then it is known as *graph labeling*.

For latest survey on graph labeling we refer to J. A. Gallian[3]. Vast amount of literature is available on different types of graph labeling and more than 1000 research papers have been published so far in last four decades.

The present work is aimed to discuss one such labeling known as prime cordial labeling.

Definition 1.2: A mapping $f: V(G) \to \{0,1\}$ is called binary vertex labeling of G and $f(v)$ is called the label of the vertex v of G under f.

Definition 1.3: If for an edge $e=uv$, the induced edge labeling $f^*: E(G) \to \{0,1\}$ is given by $f^*(e) = |f(u) - f(v)|$. Here

$$\left.\begin{array}{l} v_f(i) = \text{Number of vertices of } G \text{ having label } i \text{ under } f \\ e_f(i) = \text{Number of edges of } G \text{ having label } i \text{ under } f^* \end{array}\right\}$$

where $i = 0$ or 1

Definition 1.4: A *prime cordial labeling* of a graph G with vertex set V is a bijection f from V to $\{1,2,3,\ldots,|V|\}$ such that if each edge uv is assigned the label 1 if $gcd(f(v), f(u))=1$ and 0 if $gcd(f(v), f(u))>1$, then the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. A graph which admits prime cordial labeling is said to be a *prime cordial graph*.

The notion of prime cordial labeling was introduced by Sundaram, Ponraj and Somasundaram[4]. In this paper they have proved that C_n is a prime cordial if and only if $n \geq 6$, P_n is a prime cordial if and only if $n \neq 3$ or 5. They have also proved that bistars, dragons, crowns graphs are prime cordial. In[5] Babujee and Shobana proved sun graphs $C_n \odot K_1$, C_n with a path of length $n - 3$ attached to a vertex, and P_n $(n \geq 6)$ with $n - 3$ pendent edges attached to a pendent vertex of P_n have prime cordial labelings.

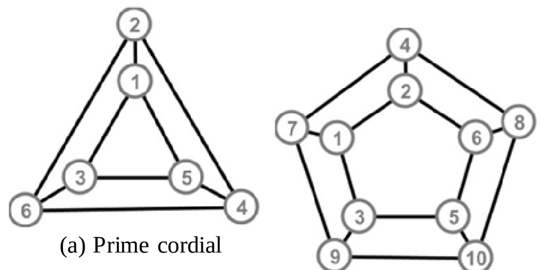
Definition 1.5: A *generalized prism graph* $Y_{m,n}$ is the graph Cartesian product $Y_{m,n}=C_n \times P\_m$.

$Y_{m,n}$ has mn vertices and $m(2n-1)$ edges. Let $V(Y_{m,n}) = \{v_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E(Y_{m,n}) = \{v_{ij}\, v_{(i+1)j} : 1 \leq i \leq m-1, 1 \leq j \leq n\} \cup \{v_{mj}v_{1j} : 1 \leq j \leq n\} \cup \{v_{ij}v_{i(j+1)} : 1 \leq i \leq m, 1 \leq j \leq n-1\}$.

2. Prime Cordial Labeling of Generalized Prism graph $Y_{m,n}$:

Theorem 2.1: $Y_{2n+1,2}$ is prime cordial for all n.

Proof: In the following Figure 1, we have shown a prime cordial labeling of $Y_{3,2}$ and a prime cordial labeling of $Y_{5,2}$.



(a) Prime cordial labeling of $Y_{3,2}$

(b) Prime cordial labeling of $Y_{5,2}$

Figure 1: Prime cordial labeling of $Y_{3,2}$ and $Y_{5,2}$

From the above Figures we can check that for the graph $Y_{3,2}$, $|e_f(1) - e_f(0)| = |5-4| = 1$ and for the graph $Y_{5,2}$, $|e_f(1) - e_f(0)| = |7-8| = 1$.

Therefore both the graphs $Y_{3,2}$ and $Y_{5,2}$ are prime cordial.

Now for the graph $Y_{2n+1,2}$ for $n > 2$, first of all take the graph $Y_{5,2}$ with prime cordial labeling shown in Figure 1(b). Then add n-2 vertices on each edge joining the vertices with label 1 and 2, 2 and 6, 7 and 4, 4 and 8 as shown in the following Figure 2. Then label these new added $4n$-8 vertices as follows:

- Label the n - 2 vertices on the edge joining the vertices with label 1 and 2 by the numbers 11, 15, 19,..., $4n$ - 5, $4n$ - 1 in clockwise direction as shown in the following Figure 2.
- Label the n - 2 vertices on the edge joining the vertices with label 7 and 4 by the numbers 13, 17, 21,..., $4n$ - 3, $4n$ + 1 in clockwise direction as shown in the following Figure 2.
- Label the n - 2 vertices on the edge joining the vertices with label 2 and 6 by the numbers 12, 16, 20,..., $4n$ - 4, $4n$ in anticlockwise direction as shown in the following Figure 2.
- Label the n - 2 vertices on the edge joining the vertices with label 4 and 8 by the numbers 14, 18, 22,..., $4n$ - 2, $4n$ + 2 in anticlockwise direction as shown in the following Figure 2.

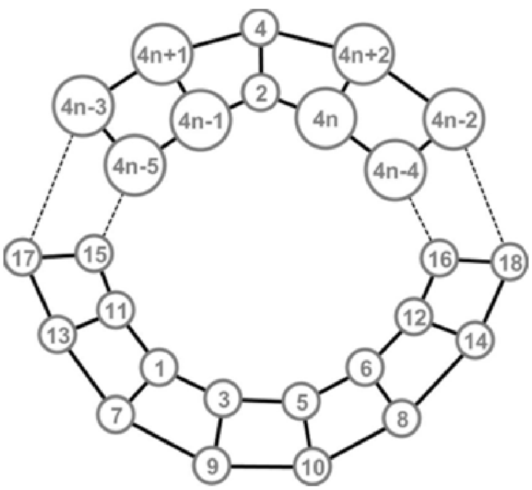


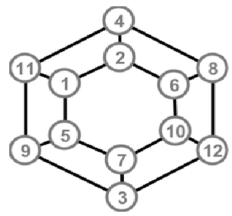
Figure 2. Prime cordial labeling of $Y_{2n+1,2}$

From the above Figure 2, we can easily check that all the new edges added to the left side to the vertices with label 2 and 4 have label 1 and to the right side to the vertices with label 2 and 4 have label 0. So the difference of $e_f(0)$ and $e_f(0)$ remains same as in $Y_{5,2}$ which is 1. So for the graph $Y_{2n+1,2}$, $|e_f(1) - e_f(0)| = 1$.

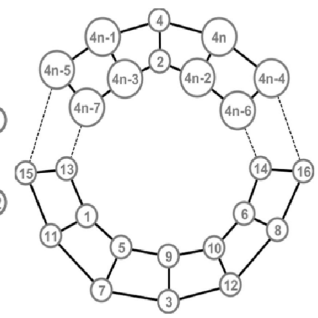
Thus the graph $Y_{2n+1,2}$ is prime cordial for all n.

Theorem 2.2: $Y_{2n,2}$ is prime cordial for all $n > 2$.

Proof: In the following Figure 3(a), we have shown the prime cordial labeling of $Y_{6,2}$.



(a) Prime cordial labeling of $Y_{6,2}$



(b) Prime cordial labeling of $Y_{2n,2}$

Figure 3. Prime cordial labeling of $Y_{6,2}$ and $Y_{2n,2}$

From the above Figure 3(a), we can check that for the graph $Y_{6,2}$, $|e_f(1)-e_f(0)|=|9-9|=0$. Therefore the graph $Y_{6,2}$ is prime cordial.

Now for the graph $Y_{2n,2}$ for $n>3$, first of all take the graph $Y_{6,2}$ with prime cordial labeling shown in the Figure 3(a). Then add n-3 vertices on each edge joining the vertices with label 1 and 2, 2 and 6, 11 and 4, 4 and 8 as shown in the previous Figure 3(b). Then label these new added $4n$ - 12 vertices as follows:

- Label the n - 3 vertices on the edge joining the vertices with label 1 and 2 by the numbers 13, 17, 21,..., $4n$ -7, $4n$ - 3 in clockwise direction as shown in the above Figure 3(b).
- Label the n - 3 vertices on the edge joining the vertices with label 11 and 4 by the numbers 15, 19, 26,..., $4n$ - 5, $4n$ - 1 in clockwise direction as shown in the above Figure 3(b).
- Label the n - 3 vertices on the edge joining the vertices with label 2 and 6 by the numbers 14, 18, 22,..., $4n$ - 6, $4n$ - 2 in anticlockwise direction as shown in the above Figure 3(b).
- Label the n - 3 vertices on the edge joining the vertices with label 4 and 8 by the numbers 16, 20, 24,..., $4n$ - 4, $4n$ in anticlockwise direction as shown in the above Figure 3(b).

From the previous Figure 3(b) we can easily check that all the new edges added to the left side to the vertices with label 2 and 4 have label 1 and to the right side to the vertices with label 2 and 4 have label 0. So the difference of $e_f(1)$ and $e_f(0)$ remains same as in $Y_{6,2}$ which is 0. So for the graph $Y_{2n,2}$,$|e_f(1)$-$e_f(0)|$=0.

Thus the graph $Y_{2n,2}$ is prime cordial for all $n > 2$.

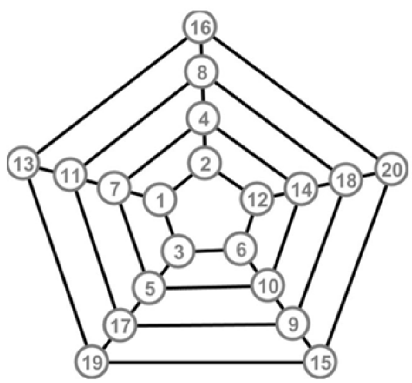
Note: Graph $Y_{4,2}$ is not prime cordial, which can be easily verified.

From the Theorem 2.1 and Theorem 2.2, we have the following result:

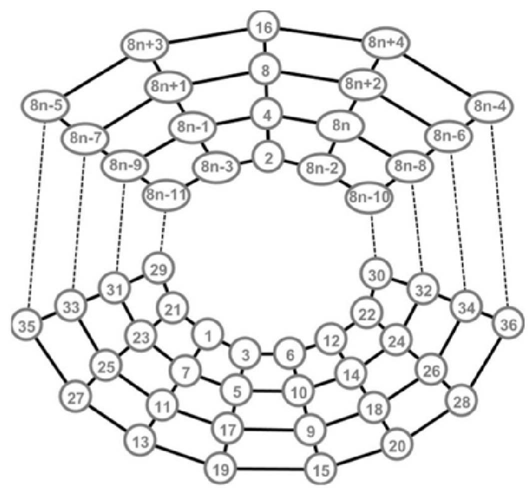
Theorem 2.3: $Y_{n,2}$ is prime cordial for all n except n = 1, 2 and 4.

Theorem 2.4: $Y_{2n+1,4}$ is prime cordial for all n.

Proof: In the Theorem 2.1, we have proved that the graph $Y_{3,4}$ is prime cordial. Now in the following Figure 4(a), we have shown the prime cordial labeling of $Y_{5,4}$.



(a) Prime cordial labeling of $Y_{5,4}$



(b) Prime cordial labeling of $Y_{2n+1,4}$

Figure 5. Prime cordial labeling of $Y_{5,4}$. and $Y_{2n+1,4}$

From the previous Figure 4(a), we can check that for the graph $Y_{5,4}$,$|e_f(1) - e_f(0)|=|18-17|=1$. Therefore both the graphs $Y_{3,4}$ and $Y_{5,4}$ are prime cordial.

Now for the graph $Y_{2n+1,4}$ for $n > 2$, first of all take the graph $Y_{5,4}$ with prime cordial labeling shown in previous Figure 4(a). Then add $2n$-4 vertices on each edge joining the vertices with label 1 and 2, 2 and 12, 7 and 4, 4 and 14, 11 and 8, 8 and 18, 13 and 16, 16 and 20 as shown in the previous Figure 4(b). Then label these new added 8n - 16 vertices as follows:

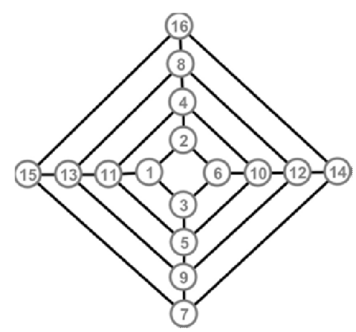
- Label the $2n$ - 4 vertices on the edge joining the vertices with label 1 and 2 by the numbers 21, 29, 37,..., $8n$ - 11, $8n$ - 3 in clockwise direction as shown in the previous Figure 4(b).
- Label the $2n$ - 4 vertices on the edge joining the vertices with label 7 and 8 by the numbers 23, 31, 39,..., $8n$ - 9, $8n$ - 1 in clockwise direction as shown in the previous Figure 4(b).
- Label the $2n$ - 4 vertices on the edge joining the vertices with label 11 and 8 by the numbers 25, 33, 41,..., $8n$ - 7, $8n$ + 1 in clockwise direction as shown in the previous Figure 4(b).
- Label the $2n$ - 4 vertices on the edge joining the vertices with label 13 and 16 by the numbers 27, 35, 41,..., $8n$ - 5, $8n$ + 3 in clockwise direction as shown in the previous Figure 4(b).
- Label the $2n$ - 4 vertices on the edge joining the vertices with label 2 and 12 by the numbers 22, 30, 38,..., $8n$ - 10, $8n$ - 2 in anticlockwise direction as shown in the previous Figure 4(b).
- Label the $2n$ - 4 vertices on the edge joining the vertices with label 4 and 14 by the numbers 24, 32, 40,..., $8n$ - 8, $8n$ in anticlockwise direction as shown in the previous Figure 4(b).
- Label the $2n$ - 4 vertices on the edge joining the vertices with label 8 and 18 by the numbers 26, 34, 42,..., $8n$ - 6, $8n$ + 2 in anticlockwise direction as shown in the previous Figure 4(b).
- Label the $2n$ - 4 vertices on the edge joining the vertices with label 16 and 20 by the numbers 28, 36, 37,..., $8n$ - 4, $8n$ + 4 in anticlockwise direction as shown in the previous Figure 4(b).

From the previous Figure 4(b), we can easily check that all the new edges added to the left side to the vertices with label 2,4,8 and 16 have label 1 and to the right side to the vertices with label 2,4,8 and 16 have label 0. So the difference of e_f (1) and e_f (0) remains same as in $Y_{5,4}$ which is 1. So for the graph $Y_{2n+1,4}$, $|e_f (1) - e_f (0)|=1$.

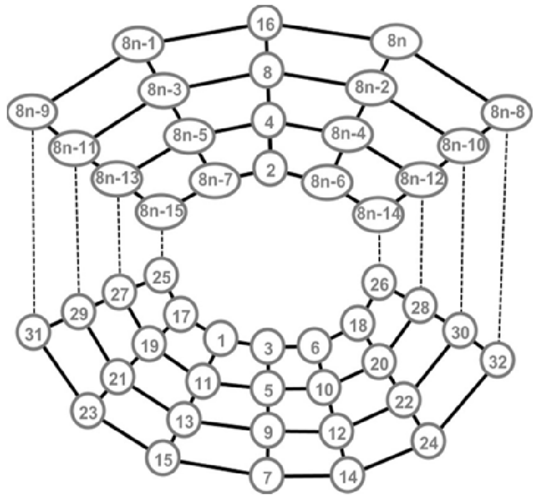
Thus the graph $Y_{2n+1,4}$ is prime cordial for all n.

Theorem 2.5: $Y_{2n,4}$ is prime cordial for all $n \geq 2$.

Proof: In the following Figure 5(a), we have shown the prime cordial labeling of $Y_{4,4}$.



(a) Prime cordial labeling of $Y_{4,4}$



(b) Prime cordial labeling of $Y_{2n,4}$

Figure 5. Prime cordial labeling of $Y_{4,4}$ and $Y_{2n,4}$

From the previous Figure 5(a), we can check that for the graph $Y_{4,4}$, $|e_f(1) - e_f(0)|=|14-14|=0$. Therefore the graphs $Y_{4,4}$ is prime cordial.

Now for the graph $Y_{2n,4}$ for $n > 2$, first of all take the graph $Y_{4,4}$, with prime cordial labeling shown in the previous Figure 5(a). Then add $2n - 4$ vertices on each edge joining the vertices with label 1 and 2, 2 and 6, 11 and 4, 4 and 10, 13 and 8, 8 and 12, 15 and 16, 16 and 14 as shown in the previous Figure 5(b). Then label these new added $8n - 16$ vertices as follows:

- Label the $2n -$ 4 vertices on the edge joining the vertices with label 1 and 2 by the numbers 17,25,33,...,$8n$ - 15, $8n$ - 7 in clockwise direction as shown in the previous Figure 5(b).
- Label the $2n -$ 4 vertices on the edge joining the vertices with label and by the numbers 19,27,35,...,$8n$ - 13, $8n$ - 5 in clockwise direction as shown in the previous Figure 5(b).
- Label the $2n -$ 4 vertices on the edge joining the vertices with label and by the numbers 21,29,37,...,$8n$ - 11, $8n$ - 3 in clockwise direction as shown in the previous Figure 5(b).
- Label the $2n -$ 4 vertices on the edge joining the vertices with label and by the numbers 23,31,39,...,$8n$ - 9, $8n$ - 1 in clockwise direction as shown in the previous Figure 5(b).
- Label the $2n -$ 4 vertices on the edge joining the vertices with label and by the numbers 18,26,34,...,$8n$ - 14, $8n$ - 6 in anticlockwise direction as shown in the previous Figure 5(b).
- Label the $2n -$ 4 vertices on the edge joining the vertices with label and by the numbers 20,28,36,...,$8n$ - 12, $8n$ - 4 in anticlockwise direction as shown in the previous Figure 5(b).
- Label the $2n -$ 4 vertices on the edge joining the vertices with label and by the numbers 22,30,38,...,$8n$ - 10, $8n$ - 2 in anticlockwise direction as shown in the previous Figure 5(b).
- Label the $2n -$ 4 vertices on the edge joining the vertices with label and by the numbers 24,32,40,...,$8n$ - 8, $8n$ in anticlockwise direction as shown in the previous Figure 5(b).

From the previous Figure 5(b), we can easily check that all the new edges added to the left side to the vertices with label 2, 4, 8 and 16 have label 1 and to the right side to the vertices with label 2, 4, 8 and 16 have label 0. So the difference of $e_f(1)$ and $e_f(0)$ remains same as in $Y_{4,4}$ which is 0. So for the graph $Y_{2n,4}$,$| e_f(1)-e_f(0)|=0$.

Thus the graph $Y_{2n,4}$ is prime cordial for all $n \geq 2$.

From the Theorem 2.4 and Theorem 2.5, we have the following result:

Theorem 2.6: $Y_{n,4}$ is prime cordial for all n.

Theorem 2.7: $Y_{3,2n}$ is prime cordial.

Proof: Here total $6n$ vertices and $3(4n - 1)$ edges in the graph $Y_{3,2n}$. Now we label these vertices as follows:

Step 1: Draw the graph $Y_{3,2}$ and label the vertices as shown in the following Figure 6. Here the vertices of inner cycle are labeled in anticlockwise direction and the vertices of outer cycle are labeled in clockwise direction.

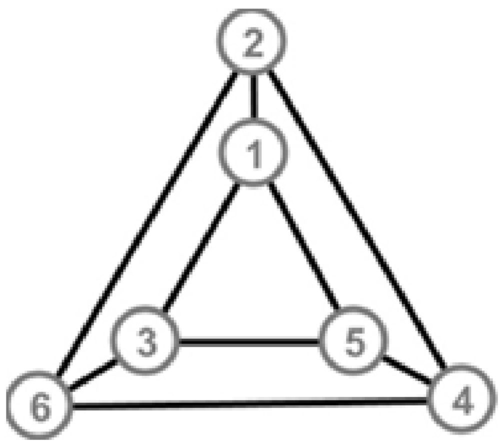


Figure 6. Prime cordial labeling of $Y_{3,2}$

Here $e_f(1) = 5$ and $e_f(0) = 4$. $\therefore$ $|e_f(1) - e_f(0)| = 1$

Step 2: Now draw one cycle inner side and one cycle outer side to the above graph and join the vertices of these cycles to the corresponding vertices of previous cycles as shown in the following Figure 7. Then label the vertices of inner cycle by 7,9,11 in clockwise direction and the vertices of outer cycle by 8, 10, 12 in clockwise direction as shown in the following Figure 7.

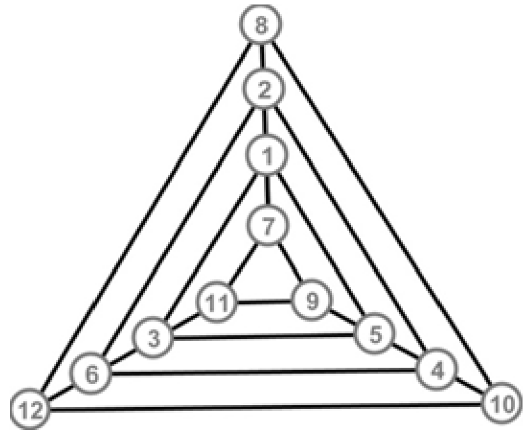


Figure 7. Prime cordial labeling of $Y_{3,4}$

Here we have added 6 vertices and 12 edges to the graph $Y_{3,2}$. Out of these 12 edges, 6 inner edges have label 1 and 6 outer edges have label 0. So the difference of $e_f(1)$ and $e_f(0)$ remains same.

Thus in k^{th} step we add one inner cycle and one outer cycle to the previous graph and label the vertices of outer cycle by next three consecutive even integers $6k - 4$, $6k - 2$, $6k$ in clockwise direction and label the vertices of inner cycle by next three consecutive odd integers in clockwise direction if k is even and in anticlockwise direction if k is odd. Thus every time 6 edges added to outer side have label 0. Now we will prove that every time 6 edges added to the inner side have label 1.

Case 1: If k is even then the vertices of inner cycle have label $6k$ - 5, $6k$ - 3, $6k$ - 1 in clockwise direction and the vertices of the previous inner cycle have label $6k$ - 11, $6k$ - 9, $6k$ - 7 in anticlockwise direction. Thus we have

$$gcd(6k-5, 6k-11) = gcd(6, 6k-11) = 1,$$
$$gcd(6k-3, 6k-7) = gcd(4, 6k-7) = 1,$$
$$gcd(6k-1, 6k-9) = gcd(8, 6k-9) = 1,$$

and $gcd(6k-5, 6k-3) = gcd(6k-3, 6k-1) = gcd(6k-1, 6k-5) = 1.$

Thus these 6 edges have label 1.

Case 2: If k is odd then the vertices of inner cycle have label $6k$ - 5, $6k$ - 3, $6k$ - 1 in anticlockwise direction and the vertices of the previous inner cycle have label $6k$ - 11, $6k$ - 9, $6k$ - 7 in clockwise direction. Thus as in case 1 we have all 6 edges with label 1.

Thus at each step the difference of $e_f(1)$ and $e_f(0)$ remains same. In the first step, this difference is 1 and so after n steps also $|e_f(1)-e_f(0)|=1$. Therefore $Y_{3,2n}$ is prime cordial.

Theorem 2.8: $Y_{3,2n+1}$ is prime cordial.

Proof: Here total $6n + 3$ vertices and $3(4n + 1)$ edges in the graph $Y_{3,2n+1}$. Now we label these vertices as follows:

Step 1: Draw the graph $Y_{3,3}$ and label the vertices as shown in the following Figure 8.

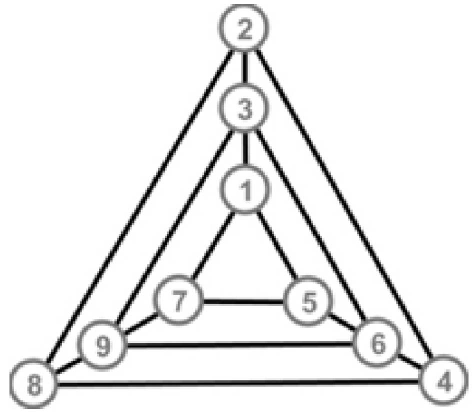


Figure 8. Prime cordial labeling of $Y_{3,3}$

Here $ef(1) = 8$ and $ef(0) = 7$. $\therefore$ $|e_f(1) - e_f(0)| = 1$

Step 2: Now draw one cycle inner side and one cycle outer side to the above graph and join the vertices of these cycles to the corresponding vertices of previous cycles as shown in the following Figure 9. Then label the vertices of inner cycle by 11, 13, 15 in clockwise direction and the vertices of outer cycle by 10, 12, 14 in clockwise direction as shown in the following Figure 9.

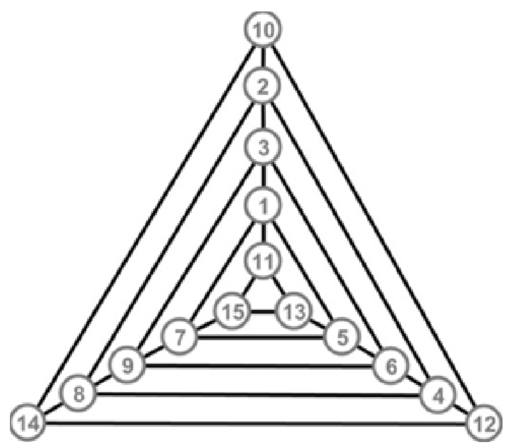


Figure 9. Prime cordial labeling of $Y_{3,5}$

Here we have added 6 vertices and 12 edges to the graph $Y_{3,3}$. Out of these 12 edges, 6 inner edges have label 1 and 6 outer edges have label 0. So the difference of $e_f(1)$ and $e_f(0)$ remains same.

Thus in k^{th} step we add one inner cycle and one outer cycle to the previous graph and label the vertices of outer cycle by next three consecutive even integers $6k$ - 2, $6k$, $6k$ + 2 in clockwise direction and label the vertices of inner cycle by next three consecutive odd integers in clockwise direction if k is even and in anticlockwise direction if k is odd. Thus every time *6* edges added to outer side have label 0. Now we will prove that every time 6 edges added to the inner side have label.

Case 1: If k is even then the vertices of inner cycle have label 6k - 1, 6k + 1, 6k + 3 in clockwise direction and the vertices of the previous inner cycle have label 6k - 7, 6k - 5, 6k - 3 in anticlockwise direction for k > 2. Thus we have

$$gcd(6k-1, 6k-7) = gcd(6, 6k-7) = 1,$$

$$gcd(6k+1, 6k-3) = gcd(4, 6k-3) = 1,$$

$$gcd(6k+3, 6k-5) = gcd(8, 6k-5) = 1,$$

and $gcd(6k-1, 6k+1) = gcd(6k+1, 6k+3) = gcd(6k+3, 6k-1) = 1.$

Thus these 6 edges have label 1.

Case 2: If k is odd then the vertices of inner cycle have label $6k$ -1, $6k$ + 1, $6k$ + 3 in anti-clockwise direction and the vertices of the previous inner cycle have label $6k$ - 7, $6k$ - 5, $6k$ + 3 in clockwise direction for k > 2. Thus as in case 1 we have all 6 edges with label 1.

Thus at each step the difference of $e_f(1)$ and $e_f(0)$ remains same. In the first step, this difference 1 and so after n steps also $|e_f(1)-e_f(0)|=1$. Therefore $Y_{3,2n+1}$ is prime cordial.

Hence from the Theorem 2:4 and Theorem 2:5, we have the following result:

Theorem 2.9: $Y_{3,n}$ is prime cordial for all n greater than 1.

Theorem 2.10: $Y_{4,2n}$ is prime cordial for all n greater than 1.

Proof: Here total $8n$ vertices and $4(4n - 1)$ edges in the graph $Y_{4,2n}$. Now we label these vertices as follows:

Step 1: Draw the graph $Y_{4,4}$ and label the vertices as shown in the following Figure 10.

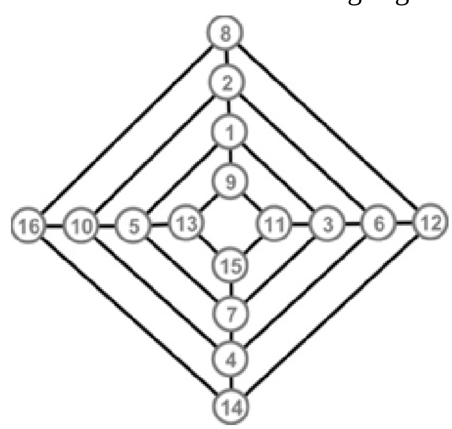


Figure 10. Prime cordial labeling of $Y_{4,4}$

Here $ef(1) = 14$ and $ef(0) = 14$.

$\therefore\ |e_f(1) - e_f(0)| = 0$

Step 2: Now draw one cycle inner side and one cycle outer side to the above graph and join the vertices of these cycles to the corresponding vertices of previous cycles as shown in the following Figure 11. Then label the vertices of inner cycle by 17,19,23,21 in clockwise direction and the vertices of outer cycle by 18,20,22,24 in clockwise direction as shown in the following Figure 11.

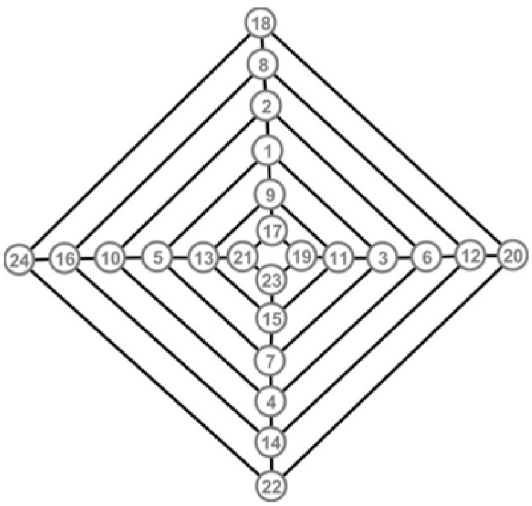


Figure 11. Prime cordial labeling of $Y_{4,6}$

Here we have added 8 vertices and 16 edges to the graph $Y_{4,4}$. Out of these 16 edges, 8 inner edges have label 1 and 8 outer edges have label 0. So the difference of $e_f(1)$ and $e_f(0)$ remains same.

Thus in k^{th} step we add one inner cycle and one outer cycle to the previous graph and label the vertices of outer cycle by next four consecutive even integers $8k$+2, $8k$+4, $8k$+6, $8k$+8 in clockwise direction and label the vertices of inner cycle by next four consecutive odd integers $8k$+1, $8k$+3, $8k$+7, $8k$+5 in clockwise direction for k>1. Thus every time 8 edges added to outer side have label 0. Now we will prove that every time 8 edges added to the inner side have label 1.

Here in k^{th} step, the vertices of inner cycle have label $8k$+1, $8k$+3, $8k$+7, $8k$+5 in clockwise direction and the vertices of the previous inner cycle have also label $8k$-7, $8k$-5, $8k$-1, $8k$-3 in clockwise direction for $k > 2$. Thus we have

$$gcd(8k+1, 8k-7) = gcd(8, 8k-7) = 1,$$
$$gcd(8k+3, 8k-5) = gcd(8, 8k-5) = 1,$$
$$gcd(8k+7, 8k-1) = gcd(8, 8k-1) = 1,$$
$$gcd(8k+5, 8k-3) = gcd(8, 8k-3) = 1,$$

and $gcd(8k+1, 8k+3) = gcd(8k+3, 8k+7) = gcd(8k+7, 8k+5) = gcd(8k+5, 8k+1) = 1$.

Thus these 8 edges have label 1.

Thus at each step the difference of $e_f(1)$ and $e_f(0)$ remains same. In the first step, this difference is 0 and so after n steps also $|e_f(1)-e_f(0)|=0$. Therefore $Y_{4,2n}$ is prime cordial for $n > 1$.

Theorem 2.11: $Y_{4,2n+1}$ is prime cordial.

Proof: Here total $8n$+4 vertices and $4(4n+1)$ edges in the graph $Y_{4,2n+1}$. Now we label these vertices as follows:

Step 1: Draw the graph $Y_{4,3}$ and label the vertices as shown in the following Figure 12.

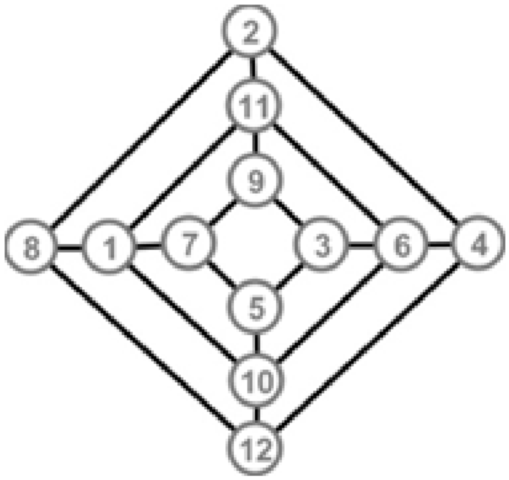


Figure 12. Prime cordial labeling of $Y_{4,3}$

Here $e_f(1)$=10 and $e_f(0)$=10. ∴ $|e_f(1)$-$e_f(1)|$=0

Step 2: Now draw one cycle inner side and one cycle outer side to the above graph and join the vertices of these cycles to the corresponding vertices of previous cycles as shown in the following Figure 13. Then label the vertices of inner cycle by 19,17,13,15 in clockwise direction and the vertices of outer cycle by 14, 16,18,20 also in clockwise direction as shown in the Figure 13.

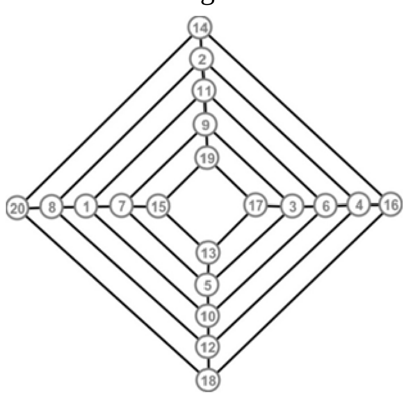


Figure 13. Prime cordial labeling of $Y_{4,5}$

Here we have added 8 vertices and 16 edges to the graph $Y_{4,3}$. Out of these 16 edges, 8 inner edges have label 1 and 8 outer edges have label 0. So the difference of $e_f(1)$ and $e_f(0)$ remains same.

Thus in k^{th} step we add one inner cycle and one outer cycle to the previous graph and label the vertices of outer cycle by next four consecutive even integers 8k-2, 8k, 8k+2, 8k+4 in clockwise direction and label the vertices of inner cycle by next four consecutive odd integers 8k-3, 8k+1, 8k-3, 8k-1 in clockwise direction for $k > 1$. Thus every time 8 edges added to outer side have label 0. Now we will prove that every time 8 edges added to the inner side have label 1.

Here in k^{th} step, the vertices of inner cycle have label 8k+3, 8k+1, 8k-3, 8k-1 in clockwise direction and the vertices of the previous inner cycle have also label 8k-5, 8k-7, 8k-11, 8k-9 in clockwise direction for $k > 2$. Thus we have

$$gcd(8k+3, 8k-5) = gcd(8, 8k-5) = 1,$$
$$gcd(8k+1, 8k-7) = gcd(8, 8k-7) = 1,$$
$$gcd(8k-3, 8k-11) = gcd(8, 8k-11) = 1,$$
$$gcd(8k-1, 8k-9) = gcd(8, 8k-9) = 1$$

and $gcd(8k+3, 8k+1) = gcd(8k+1, 8k-3) = gcd(8k-3, 8k-1) = gcd(8k-1, 8k+3) = 1.$

Thus these 8 edges have label 1.

Thus at each step the difference of $e_f(1)$ and $e_f(0)$ remains same. In the first step, this difference is 0 and so after n steps also $|e_f(1)$-$e_f(0)|$=0. Therefore $Y_{4,2n+1}$ is prime cordial.

Hence from the Theorem 2.7 and Theorem 2.8 we have the following result:

Theorem 2.12: $Y_{4,n}$ is prime cordial for all n greater than 2.

Theorem 2.13: $Y_{5,n}$ is prime cordial for all n greater than 1.

Proof: First of all draw a cycle C_5 and label the vertices as shown in the following Figure 14:

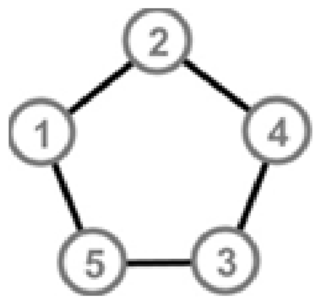
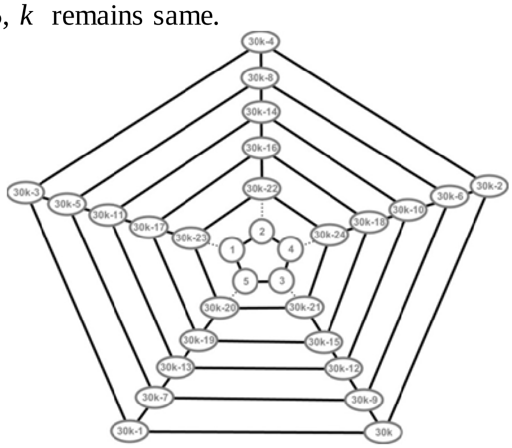
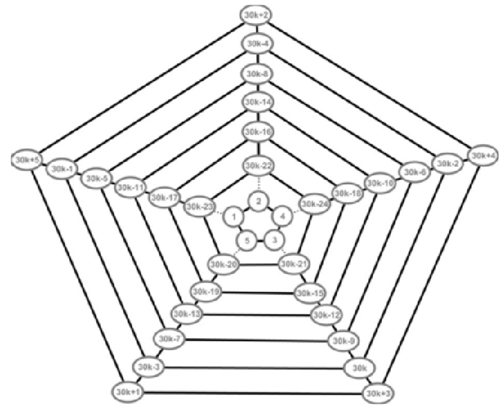


Figure 14

Now for m^{th} ($m \geq 2$) cycle of $Y_{5,n}$, take $k = \left\lceil \frac{m-2}{6} \right\rceil + 1$. Here we can easily check that for m, $m+1$, $m+2$, $m+3$, $m+4$ and $m+5$, k remains same.



(a)



(b)

Figure 15. Prime cordial labeling of $Y_{5,m}$, $Y_{5,m+1}$, $Y_{5,m+2}$, $Y_{5,m+3}$, $Y_{5,m+4}$ and $Y_{5,m+5}$

Now

Step 1: Label the vertices of m^{th} cycle of $Y_{5,n}$ by the integers 30k-22, 30k-24, 30k-21, 30k-20, 30k-23 in clockwise direction as shown in the previous Figure 15(a).

Step 2: Label the vertices of $(m+1)^{th}$ cycle by the integers 30k-16, 30k-18, 30k-15, 30k-19, 30k-17 in clockwise direction as shown in the previous Figure 15(a). By adding $(m+1)^{th}$ cycle, we are adding 10 edges to the graph $Y_{5,m}$. Now for these 10 edges, we have $gcd(30k-15, 30k-19) = gcd(30k-19, 30k-17) = gcd(30k-17, 30k-16) = gcd(30k-19, 30k-20) = gcd(30k-17, 30k-23) = 1$ and other five edges have gcd greater than 1. Thus out of these 10 edges, 5 edges have label 1 and other 5 edges have label 0. So if the graph $Y_{5,m}$ is prime cordial then the graph $Y_{5,m+1}$ is also prime cordial.

Step 3: Label the vertices of $(m+2)^{th}$ cycle by the integers 30k-14, 30k-10, 30k-12, 30k-13, 30k-11 in clockwise direction as shown in the previous Figure 15(a). By adding $(m+2)^{th}$ cycle, we are adding 10 edges to the graph $Y_{5,m+1}$. Now for these 10 edges, we have $gcd(30k-12, 30k-13) = gcd(30k-13, 30k-11) = gcd(30k-11, 30k-14) = gcd(30k-13, 30k-19) = gcd(30k-11, 30k-17) = 1$ and other five edges have gcd greater than 1. Thus out of these 10 edges, 5 edges have label 1 and other 5 edges have label 0. So if the graph $Y_{5,m+1}$ is prime cordial then the graph $Y_{5,m+2}$ is also prime cordial.

Step 4: Label the vertices of $(m+3)^{th}$ cycle by the integers 30k-8, 30k-6, 30k-9, 30k-7, 30k-5 in clockwise direction as shown in the previous Figure 15(a). By adding cycle $(m+3)^{th}$ cycle we are adding 10 edges to the graph $Y_{5,m+2}$. Now for these 10 edges, we have $gcd(30k-9, 30k-7) = gcd(30k-7, 30k-5) = gcd(30k-5, 30k-8) = gcd(30k-7, 30k-13) = gcd(30k-5, 30k-11) = 1$ and other five edges have gcd greater than 1. Thus out of these 10 edges, 5 edges have label 1 and other 5 edges have label 0. So if the graph $Y_{5,m+2}$ is prime cordial then the graph $Y_{5,m+3}$ is also prime cordial.

Step 5: Label the vertices of $(m+4)^{th}$ cycle by the integers 30k-4, 30k-2, 30k, 30k-1, 30k-3 in clockwise direction as shown in the previous Figure 15(a). By adding $(m+4)^{th}$ cycle, we are adding 10 edges to the graph $Y_{5,m+3}$. Now for these 10 edges, we have $gcd(30k, 30k-1) = gcd(30k-1, 30k-3) = gcd(30k-3, 30k-4) = gcd(30k-1, 30k-7) = gcd(30k-3, 30k-5) = 1$ and other five edges have gcd greater than 1. Thus out of these 10 edges, 5 edges have label 1 and other 5 edges have label 0. So if the graph $Y_{5,m+3}$ is prime cordial then the graph $Y_{5,m+4}$ is also prime cordial.

Step 6: Label the vertices of $(m+5)^{th}$ cycle by the integers 30k+2, 30k+4, 30k+3, 30k+1, 30k+5 in clockwise direction as shown in the previous Figure 15(b). By adding $(m+5)^{th}$ cycle, we are adding 10 edges to the graph $Y_{5,m+4}$. Now for these 10 edges, we have $gcd(30k+4, 30k+3) = gcd(30k+3, 30k+1) = gcd(30k+1, 30k+5) = gcd(30k+5, 30k+2) = gcd(30k+1, 30k-1) = gcd(30k+5, 30k-3) = 1$ and other four edges have gcd greater than 1. Thus out of these 10 edges, 6 edges have label 1 and other 4 edges have label 0. Now interchange the labels 30k-1 and 30k-3 of the $(m+4)^{th}$ cycle. As $gcd(30k, 30k-3) > 1$, number of edges label with 0 increase by 1. So if the graph $Y_{5,m+4}$ is prime cordial then the graph $Y_{5,m+5}$ is also prime cordial.

Now for the $(m+5)^{th}$, $(m+7)^{th}$, $(m+8)^{th}$,

$(m+9)^{th}$, $(m+10)^{th}$, and $(m+11)^{th}$ cycles, repeat the above steps 1 to 6 respectively by taking $k+1$ instead of k.

So the vertices of $(m+11)^{th}$ cycle are labeled by the integers $30k+8$, $30k+6$, $30k+9$, $30k+10$, $30k+7$. By adding $(m+11)^{th}$ cycle, we are adding 10 edges to the graph $Y_{5,m+5}$. Now for these 10 edges, we have $gcd(30k+9, 30k+10) = gcd(30k+10, 30k+7) = gcd(30k+7, 30k+8) = gcd(30k+10, 30k+1) = gcd(30k+7, 30k+5) = 1$ and other five edges have gcd greater than 1.

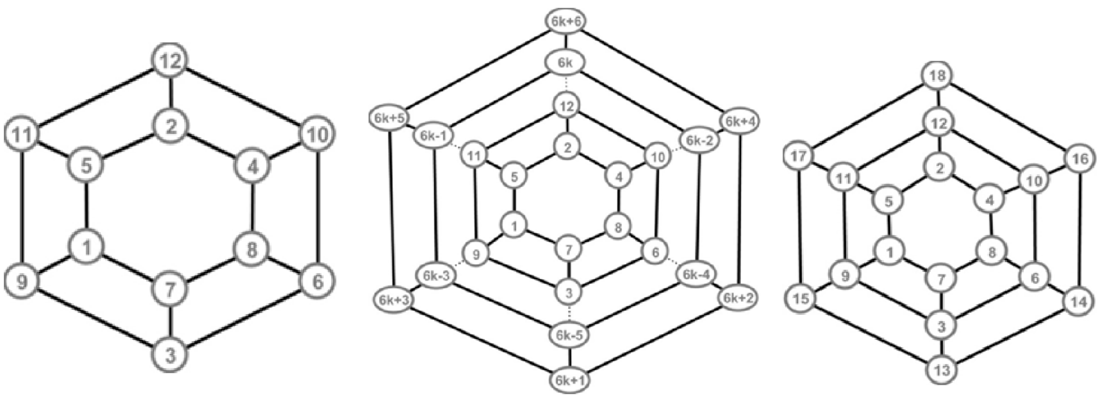
Thus out of these 10 edges, 5 edges have label 1 and other 5 edges have label 0. So if the graph $Y_{5,m+5}$ is prime cordial then the graph $Y_{5,m+6}$ is also prime cordial. And we can easily check that the graph $Y_{5,2}$ is prime cordial by taking $m = 2$ in step 1.

Thus $Y_{5,n}$ is prime cordial for all n greater than 1.

Theorem 2.14: $Y_{6,2}$ is prime cordial for all n greater than 1.

Proof: Draw the graph $Y_{6,2}$ and label the vertices as shown in the following Figure 16(a):

From the following Figure 16(a), we can easily check that the graph $Y_{6,2}$ is prime cordial. Now for the graph $Y_{6,k}$ ($k \geq 3$), label the vertices of k^{th} cycle by the integers $6k$, $6k$-2, $6k$-4, $6k$-5, $6k$-3, $6k$-1 in clockwise direction as shown in the following Figure 16(b). We can easily check that for $k = 3$, the graph $Y_{6,3}$ which is shown in the Figure 16(c) is prime cordial. Now for the graph $Y_{6,k+1}$, the vertices of the $(k+1)^{th}$ cycle are labeled by the integers $6k+6$, $6k+4$, $6k+2$, $6k+1$, $6k+3$, $6k+5$ in clockwise direction as shown in the above Figure 16(b). By adding $(k+1)^{th}$ cycle to the graph $Y_{6,k}$, we are adding 12 edges to the graph $Y_{6,k}$. Now for these 12



(a) Prime cordial labeling of $Y_{6,2}$

(b) Prime cordial labeling of $Y_{6,k}$ and $Y_{6,k+1}$

(c) Prime cordial labeling of $Y_{6,3}$

Figure 16. Prime cordial labeling of $Y_{6,n}$

edges, we have $gcd(6k+2, 6k+1) = gcd(6k+1, 6k+3) = gcd(6k+3, 6k+5) = gcd(6k+5, 6k+6) = gcd(6k+1, 6k-5) = gcd(6k+5, 6k-1) = 1$ and other six edges have gcd greater than 1.

Thus out of these 12 edges, 6 edges have label 1 and other 6 edges have label 0. So if the graph $Y_{6,k}$ is prime cordial then the graph $Y_{6,k+1}$ is also prime cordial.

Thus $Y_{6,n}$ is prime cordial for all n greater than 1.

Theorem 2.15: $Y_{2p,n}$ is prime cordial for all odd prime p and for all $n > 1$.

Proof: Prime cordial labeling of the graph $Y_{2p,2}$ is shown in the following Figure 17:

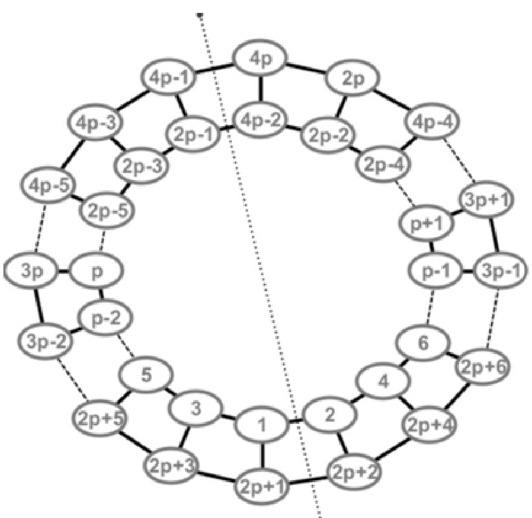


Figure 17. Prime cordial labeling of $Y_{2p,2}$

In the above Figure 17, it is seen that p vertices of the inner cycle left to the dotted line are labeled by the odd integers 1,3,5,...2p-1 and p vertices of the outer cycle left to the dotted line are labeled by the odd integers 2p+1, 2p+3, 2p+5,...,4p-1 in clockwise direction. Also p vertices of the inner cycle right to the dotted line are labeled by the even integers 2,4,6,...,2p-2, 4p-2 and p vertices of the outer cycle right to the dotted line are labeled by the odd integers 2p+2, 2p+4, 2p+6,...,4p-4, 2p, 4p in anticlockwise direction.

Now in the previous Figure 17, all 2p vertices left to the dotted line are labeled with the even integers. So they give us 3p-2 edges with label 0. Also the edge joining the vertices labeled by 2p-1 and 4p-2 and the edge joining the vertices labeled by p and 3p also have label 0. Thus we have total 3p-2+2=3p edges with label 0. And all other 3p edges have label 1 as the labels of adjacent vertices are consecutive odd integers or they differ by 2p. Thus $Y_{2p,2}$ is prime cordial.

Now for the graph $Y_{2p,n}$, label the vertices of first two inner cycles as shown in the previous Figure 17. Now for $k \geq 3$, label each vertex of k^{th} cycle (from inner side) by adding 2p to the label of the vertex of previous inner cycle adjacent to this vertex. By adding one outer cycle, we are adding 2p vertices and 4p edges to the previous graph. Out of these 4p edges, 2p edges have label 0 and other 2p edges have label 1. Because if a vertex of $(k-1)^{th}$ cycle is labeled with even integer then the corresponding vertex of k^{th} cycle is also labeled with even integer as they differ by 2p. Thus the p vertices of k^{th} cycle labeled with even integers gives us 2p - 1 edges with label 0 and one more edge joining the vertices labeled by $(2k-1)p$ and $(2k+1)p$ have also label 0. Thus we have total $2p$ - 1+1=$2p$ edges with label 0. And all other $2p$ edges have label 1 as said before that the labels of adjacent vertices are consecutive odd integers or they differ by $2p$.

Thus $Y_{2p,n}$ is prime cordial for all odd prime p and for all $n > 1$.

Conjecture 2.15: $Y_{m,n}$ is prime cordial for all $m \geq 3$ and $m \geq 2$ except $Y_{4,2}$.

3. Concluding Remarks:

Generalized prism graphs are very interesting for researchers of graph theory. It is very interesting to investigate prime cordial labeling of generalized prism graph. Here we investigate several results of generalized prism graph about prime cordial labeling. Extending the study to other results about prime cordial labeling of generalized prism graph is an open area of research.

4. Comment:

We are thankful to UGC as this work has been carried out under the Minor Research Project.

References

1. J. Gross and J. Yellen, Handbook of Graph Theory, CRC Press (2004).
2. D. M. Burton, Elementary Number Theory. 2nd ed. (1990).
3. J. A. Gallian, "A Dynamic Survey of Graph Labeling", *The Electronic Journal of Combinatorics*, #DS6, (2014).
4. M. Sundaram, R. Ponraj and S. Somasundram, "Prime cordial labeling of graphs", *J. Indian Acad. Math.*, *27,* 373-390 (2005).
5. J. Baskar, Babujee and L. Shobana, "Prime cordial labelings", *Int. Review on Pure and Appl. Math.*, 5, 277-282 (2009).