Ultra Scientist Vol. 27(3)A, 217-222 (2015).

Weakly P Properties and Not-Separation Axiomsfor Urysohn and Weakly Urysohn Axioms

CHARLES DORSETT

Department of Mathematics, Texas A&M University-Commerce
Commerce, Texas 75429

(Acceptance Date 27th October, 2015)

Abstract

In a recent paper, a new category of topological properties called weakly Po properties were introduced and investigated. The search for a topological property that failed to be a weakly Po property led to the use of "not-T_0" within that paper, and the investigation of other "not-separation axioms" and other weakly Pproperties in follow up papers. Within this paper, the study of weakly P properties and "not-separation axioms" continues with the Urysohn and weakly Urysohn axioms.

Key words: T_0-identification spaces, weakly P properties, "not-separation axioms"

Subject Classification: 54A05, 54B10, 54D10

1. Introduction

T_0-identification spaces were introduced in 1936 [11].

Definition 1.1. Let (X,T) be a space, let R be the equivalence relation on X defined by xRy iff Cl({x}) = Cl({y}), let X_0 be the set of R equivalence classes of X, let N be the nature map from X onto X_0, and let Q(X,T) be the decomposition topology on X_0 determined by (X,T) and the map N. Then (X_0,Q(X,T)) is the T_0-identification space of (X,T).

Within a 1975 paper[10], T_0-identification spaces were used to further characterize weakly Hausdorff spaces.

Theorem 1.1. A space is weakly Hausdorff iff its T_0-identification space is Hausdorff.

In the 1936 paper[11], T_0-identification spaces were used to further characterize pseudometrizable spaces.

Theorem 1.2. A space is pseudometrizable iff its T_0-identification space is metrizable.

As a result, the question of whether the process used to characterize pseudometrizable and weakly Hausdorff could be generalized to include additional topological properties arose leading to the introduction and investigation of weakly Po properties[2].

Definition 1.2. Let P and S be topological properties. A space has property P implies S iff the space is a P space that satisfies S [2].

For convenience, for a topological property P, let P implies T_0 be denoted by Po.

Definition 1.3. Let P be a topological property for which Po exists. Then (X,T) is weakly Po iff $(X_0,Q(X,T))$ has property P. A topological property Po for which weakly Po exists is called a weakly Po property[2].

Within the paper[2], it was proven that a space is weakly Po iff it T_0-identification space has property Po. Thus metrizable was the first known weakly Po property with weakly (metrizable) = pseudometrizable[11], withHausdorff added to the weakly Po properties in 1975 [10].

In the 1975 paper[10], it was proven that weakly Hausdorff is equivalent to the R_1 separation axiom, which was introduced in 1961 [1].

Definition 1.4. A space (X,T) is R_1 iff for x,y in X such that Cl({x}) and Cl({y}) are unequal, there exist disjoint open sets U and V such that x is in U and y is in V.

Thus Hausdorff is a weakly Po property with weakly (Hausdorff) = R_1.

Within the 1961 paper[1], the R_0 separation axiom was revisited and further investigated.

Definition 1.5. A space is R_0 iff for each open set O and each x in O, Cl({x}) is a subset of O.

In the paper[2], it was shown that T_1 is a weakly Po property with weakly $T_1 = R_0$ and weakly $T_2 = R_1$.. Also, within the paper[2], it was shown that for a weakly Po property Qo, a space is weakly Qo iff its T_0-identification space is weakly Qo. Combining this result with the knowledge that other properties are simultaneously shared by both a space and its T_0-identification space led to the introduction of T_0-identification P properties[3].

Definition 1.6. Let Q be a topological property. Then Q is a T_0-identification P property iff both a space and its T_0-identification space simultaneously share property Q].

In the paper[3], it was shown that for a T_0-identification P property Q, Q = weakly Qo.

Within weakly Po properties, the T_0 separation axiom has a major role raising the questions of what would happen if T_0 in the definition of weakly Po was replaced by T_1 or T_2 and leading to the introduction of weakly P1 [4] and weakly P2 [5] properties.

For a topological property P, let P1 denote P implies T_1 and let P2 denote P implies T_2.

Definition 1.7. Let P be a topological property for which P1 exists. Then a space (X,T) is weakly P1 iff $(X_0,Q(X,T))$ is P1. A topological property P1 for which weakly P1 exists is called a weakly P1 property.

Definition 1.8. Let P be a topological property for which P2 exists. Then a space (X,T) is weakly P2 iff $(X_0,Q(X,T))$ has property P2. A topological property for which weakly P2 exists is called a weakly P2 property.

Within the paper[2], the search for a topological property that failed to be weakly Po focused attention on the "not-T_0" separation axiom leading to two investigations of "not-separation axioms"; [6] and[7]. In this paper the investigation of weakly P properties and "not-separation axioms" continueswith the Urysohn and weakly Urysohn axioms.

2. More Weaky P properties:

Urysohn spaces were introduced in 1925 [12].

Definition 2.1. A space (X,T) is Urysohn iff for distinct elements x and y in X, there exist open sets U and V such that x is in U, y is in V, and Cl(U) and Cl(V) are disjoint.

In 1988, Urysohn spaces were generalized to weakly Urysohn spaces[8].

Definition 2.2. A space (X,T) is weakly Urysohn iff for x, y in X such that Cl({x}) is not Cl({y}), there exist open setsU and V such that x is in U, y is in V, and Cl(U) and Cl(V) are disjoint.

Theorem 2.1. Let (X,T) be a space. Then the following are equivalent: (a) (X,T) is Urysohn, (b) (X,T) is weakly Urysohn and T_2, (c) (X,T) is weakly Urysohn and T_1, and (d) (X,T) is weakly Urysohn and T_0.

Proof: (a) implies (b): Since (X,T) is Urysohn, then (X,T) is weakly Urysohn [ ]. Clearly (X,T) is T_2.

Clearly (b) implies (c) and (c) implies (d).

(d) implies (a): Since (X,T) is weakly Urysohn, then (X,T) is R_1 [8], which implies (X,T) is R_0 [1]. Then (X,T) is R_0 and T_0, which implies (X,T) is T_1 [1]. Thus singleton sets are closed and since (X,T) is weakly Urysohn, then (X,T) is Urysohn.

Within the 1988 paper[8], it was proven that a space is weakly Urysohn iff its T_0-identification space is Urysohn, which, when combined with the results above and the fact that for a space (X,T), $(X_0,Q(X,T))$ is T_0[ ] gives the following result.

Corollary 2.1. Urysohn is a weakly P2 property with weakly (Urysohn) = weakly Urysohn.

Within a recent paper[5], it was proven that for a weakly P2 property Q2, the least topological property P for which T_0-identification P = weakly Po = weakly P1 = weakly P2 is R_1. Combining this result with the results above give the following corollary.

Corollary 2.2. For the Urysohn property, T_0-identification (weakly Urysohn) = weakly (Urysohn)o = weakly (Urysohn)1 = weakly (Urysohn)2 = (weakly Urysohn).

In the weakly Po paper[2], topological properties which failed to be weakly Po properties were sought. Within the paper[5], it was shown that for each weakly P2 property Q2, weakly Q2 can be decomposed into two topological properties neither of which are weakly Q2 properties. The same result is known for T_0-identification P and, weakly Po [3], and weakly P1 properties[4]. Combining this result with the results above gives the next result.

Corollary 2.3. Each of T_0-identification (weakly Urysohn), weakly (Urysohn)o, weakly (Urysohn)1, and weakly (Urysohn)2 can be decomposed into the same two topological properties neither of which are weakly (Urysohn)2 properties.

In the paper[5], it was shown that for a weakly P2 property Q2, weakly Q2 is the least element of {S |S is a topological property, So exists, and So implies Q2}. A similar result is known for weakly Po [2] and weakly P1 properties[3].

Corollary 2.4. Weakly Urysohn is the least of all topological properties S for which So exists and So implies Urysohn.

Within the initial weakly P properties paper cited above[2], it was shown that both T_0 and "not-T_0" are topological properties that failed to be weakly Po properties. The work above gives many more topological properties that fail to be weakly Po properties. The role played by "not-T_0" raised questions about other "not-separation axioms" leading to two investigations of "not-separation axioms[6] and[7]. In the section below the study of "not-separation axioms" continues with the investigation of the "not-Urysohn" and "not-weakly Urysohn" axioms.

3. More "Not-Separation Axioms". Within the papers[6] and[7], "not-T_i",i = 0,1,2, and "not-R_i, i = 0, 1, were investigated. Below "not-Urysohn" and "not-weakly Urysohn" are defined and investigated.

Definition 3.1. A space (X,T) is "not-Urysohn" iff there exist distinct elements x and y such that for each open set U containing x and each open set V containing y, Cl(U) and Cl(V) are not disjoint.

Definition 3.2. A space (X,T) is "not-weakly Urysohn" iff there exist x and y in X with Cl({x}) not Cl({y}) such that for each open set U containing x and each open set V containing y, Cl(U) and Cl(V) are not disjoint.

Below natural questions concerning product spaces and subspaces of "not-weakly Urysohn" and "not-Urysohn" spaces are addressed before moving forward to resolve other questions concerning "not-weakly Urysohn" and "not- Urysohn" spaces.

Theorem 3.1. The product space (X,W), with the Tychonoff topology, of spaces $\{(X_a,T_a)$: a is in A} is "not-weakly Urysohn" iff there exists a b in A such that (X_b,T_b) is "not-weakly Urysohn".

Proof: Since (X,W) is weakly Urysohn iff for each a in A, (X_a,T_a) is weakly Urysohn[8], then, by the equivalent contrapositive statement, (X,W) is "not-weakly Urysohn" iff there exists a b in A such that (X_b,T_b) is "not-weakly Urysohn".

Since a product space is Urysohn iff each factor space is Urysohn, then "not-weakly Urysohn" in Theorem 3.1 can be replaced by "not-Urysohn".

Theorem 3.2. A space (X,T) is "not-weakly Urysohn" iff there exists a subspace (Y,T_Y) that is "not-weakly Urysohn".

Proof: Since a space (X,T) is weakly Urysohn iff each subspace of (X,T) is weakly Urysohn[8], then (X,T) is "not-weakly Urysohn" iff there exists a subspace (Y,T_Y) of (X,T) that is "not-weakly Urysohn".

Since a space is Urysohn iff each subspace is Urysohn, then "not-weakly Urysohn" in Theorem 3.2 can be replaced by "not-Urysohn".

Theorem 3.3. "Not-weakly Urysohn"

implies "not-Urysohn".

Proof: Since Urysohn implies weakly Urysohn[8], then, by the equivalent contrapositive statement, "not-weakly Urysohn" implies "not-Urysohn".

In the same manner, "not-R_1" implies "not-Urysohn" and "not-R_1" implies "not-weakly Urysohn". Since for a space(X,T) the following are equivalent: (a) (X,T) is weakly Urysohn , (b) (X_0,Q(X,T)) is Urysohn, and (c) (X_0,Q(X,T)) is weakly Urysohn, then for a space (X,T), the following are equivalent: (a) (X,T) is "not-weakly Urysohn", (b) (X_0, Q(X,T)) is "not-Urysohn", and (c) (X_0,Q(X,T)) is "not-weakly Urysohn", giving the following result.

Corollary 3.1. "Not-weakly Urysohn" is a T_0-identification P property.

Simple examples can be given showing (("not-weakly Urysohn") and T_0) need not imply (("not-weakly Urysohn") and T_1). The finite complement topology on and infinite set shows (("not-weakly Urysohn") and T_1) need not imply (("not-weakly Urysohn) and T_2). Within the 1970 book [13], an example of a T_2 space that is not Urysohn was given. Thus the properties of "not-weakly Urysohn" and "not-Urysohn" are different from those of weakly Urysohn and Urysohn.

Theorem 3.4. Let (X,T) be (("not-Urysohn") and T_0). Then (X,T) is "not-weakly Urysohn".

Proof: Let x and y be distinct elements of X such that for each open set U containing x and for each open set V containing y, Cl({U}) and Cl(V) are not disjoint. Since (X,T) is T_0, there exists an open set containing only one of x and y and Cl({x}) is not Cl({y}). Thus (X,T) is "not-weakly Urysohn".

Corollary 3.2 Let (X,T) be T_0. Then (X,T) is "not-weakly Urysohn" iff it is "not-Urysohn".

Corollary 3.3. ("Not-weakly Urysohn")o = ("not-Urysohn")o is a weakly Po property with T_0-identification ("not-weakly Urysohn") = weakly ("not-weakly Urysohn")o = weakly ("not-Urysohn")o = "not-weakly Urysohn".

Theorem 3.5. ("Not-Urysohn")o is not a weakly P1 property.

Proof: Let (X,T) be a ("not-weakly Urysohn")o space that does not imply ("not-weakly Urysohn")1. Since (X,T) is T_0 the natural map N from (X,T) onto (X_0,Q(X,T)) is a homeomorphism [9] and (X_0,Q(X,T)) is ("not-weakly Urysohn")o = ("not-Urysohn")o and not T_1.

Theorem 3.6. Let (X,T) be ("not-Urysohn")1. Then (X,T) is "not-weakly Urysohn".

Proof: Since T_1 implies T_0, (X,T) is "not- weakly Urysohn".

Corollary 3.4. Let (X,T) be T_1. Then (X,T) is ("not-weakly Urysohn")1 iff (X,T)) is ("not-Urysohn")1.

Corollary 3.5. ("Not-weakly Urysohn)1= ("not-Urysohn")1 is a weakly P1 property with weakly (not-Urysohn)1 = (("not-weakly Urysohn") and R_0).

Within the paper[4], it was proven that for each weakly P1 property Q1, Q1 is a weakly Po property with weakly (Q1)o = weakly Q1 = (weakly Qo) and R_o, giving the following result.

Corollary 3.6. ("Not-Urysohn")1 is a weakly Po property with weakly ((not-

Urysohn)1)o = weakly (not-Urysohn)1 = "not-weakly Urysohn") and R_0.

Within the paper[4], it was proven that for a weakly P2 property Q2, Q2 is a weakly P1 and weakly Po property with weakly (Q2)o = weakly (Q2)1 = weakly Q2 = (weakly Qo) and R_1, giving the next result.

Corollary 3.7, ("Not-weakly Urysohn")2 = ("not-Urysohn")2 is a weakly Po and weakly P1 property with weakly (("not-Urysohn")2)0 = weakly (("not-Urysohn")2)1 = weakly ("not-Urysohn")2 = ("not-weakly Urysohn") and R_1, and("not-weakly Urysohn")1 and ("not-weakly Urysohn")o are not a weakly P2 property,

Combining the results above give the last results in this paper.

Corollary 3.7. T_0-identification ("not-weakly Urysohn") can be decomposed into two topological properties neither of which are T_0-identificatkion (not-weakly Urysohn") nor weakly (not-weakly Urysohn")o properties and "not-weakly Urysohn" is the least of all topological properties S for which So exists and So implies ("not-Urysohn")o.

Corollary 3.8 (("Not-weakly Urysohn") and R_0) can be decomposed into two topological properties neither of which are weakly ("not-weakly Urysohn")1 propertiesand (("not-weakly Urysohn") and R_0) is the least of all topological properties S for which So exists and So implies ("not-Urysohn")1.

Corollary 3.3 (("Not-weakly Urysohn") and R_1) can be decomposed into two topological properties neither of which are weakly ("not-weakly Urysohn")2 properties and (("not-weakly Urysohn") and R_1) is the least of all topological properties S for which So exists and So implies ("not-weakly Urysohn")2.

References

1. A. Davis, Indexed Systems of Neighborhoods for General Topological Spaces, Amer. Math. Monthly, *68,* 886-893 (1961).
2. C. Dorsett, Weakly P Properties, Fundamental *J. Math. Math. Sci., 3(1),* 83-90 (2015).
3. C. Dorsett, T_0-Identification P and Weakly P Properties, accepted by Pioneer *J. Math. Math. Sci*.
4. C. Dorsett, Weakly P1, Weakly Po, and T_0-Identification P Properties, submitted.
5. C. Dorsett, Weakly P2 and Related Properties, accepted by Fundamental *J Math. Math. Sci*.
6. C. Dorsett, Not-Separation Axioms, submitted.
7. C. Dorsett, Weakly Po and T_0-Identification P Properties, and Not-Separation Axioms, submitted.
8. C. Dorsett, Generalized Urysohn Spaces, Revista Colombiana de *Math., 22,* 149-160 (1988).
9. C. Dorsett, New Characterizations of Separation Axioms, *Bull. Cal. Math. Soc., 99(1),* 37-44 (2007).
10. W. Dunham, Weakly Hausdorff Spaces, Kyungpook *Math. J., 15(1),* 41-50 (1975).
11. M. Stone, Applications of Boolean Algebras to Topology, Mat. Sb., 765-771 (1936).
12. P. Urysohn, Uber die Machtigkeit der Zusammenhangenden Mangen, Mat. Ann., *94,* 262-295 (1925).
13. S. Willard, General Topology, Addison-Wesley Pub. Co., 1970.