Ultra Scientist Vol. 27(3)A, 255-262 (2015).

Matrix Groupoids

V. RAJESWARI and P. GAJIVARADHAN

Department of Mathematics, Pachaiyappa's College,
Chennai – 600 030 (India)

(Acceptance Date 11th December, 2015)

Abstract

In this paper for the first time groupoid matrix are introduced and their properties are analysed. A matrix groupoid or a groupoid matrix is a collection of $p \times q$ matrices with entries from a groupoid (G, *) where groupoid matrix inherits the operation of G. Clearly groupoid matrix are non associative groupoids of finite order.

Key words: Groupoid matrix, Groupoid, matrix groupoid, pseudo nilpotent, pseudo zero divisors.

1. Introduction

This paper has three section. Section one is introductory in nature. Section two gives all properties satisfied by matrix groupoids which involves several new results. The final section gives the conclusions based on this study.

In this paper using the groupoids $G = \{Z_n, *, (t, u) / t, u \in Z_n\}$; matrix groupoids are constructed using the natural product $\times_n$ induced by * of G on these matrix groupoid.[3] This work is carried out in a systematic way.

2. Matrix Groupoids

Definition 2.1: Let $G = \{Z_n, *, (t, u)\}$ be a groupoid. $M=\{(a_1, \ldots, a_m) / a_i \in Z_n; 1 \leq i \leq m\}$ be the collection of all $1 \times m$ row matrices.

Define on M the product $\times$ as follows; if $x = (a_1, a_2, \ldots, a_m)$ and $y = (b_1, b_2, \ldots, b_m) \in M$ then $x \times y = (a_1, \ldots, a_m) \times (b_1, \ldots, b_m) = (a_1 * b_1, \ldots, a_m * b_m)$ and $x \times y$ M. Thus $(M, \times)$ is defined as the row matrix groupoid.

Clearly on M is defined the induced operation from G. M is non associative and non commutative groupoid known as the matrix groupoid in general.

This will be represented by the following examples.

Example 2.1: Let $M = \{(a_1, a_2, a_3) / a_i \in G = \{Z_{10}, *, *(5, 7\}, 1 \leq i \leq 3, \times\}$ be the row matrix groupoid. M is non commutative and non associative and is of finite order.

Example 2.2: Let $M = \{(a_1, a_2, \ldots, a_6) \mid a_i \in G = \{Z_{17}, *, (0, 5)\}\ 1 \leq i \leq 6, \times\}$ be the row matrix groupoid of finite order.

Example 2.3: $P = \{(a_1, a_2, \ldots, a_9) \mid a_i \in G = \{Z_{40}, (10, 10), *\}, 1 \leq i \leq 9\}$ be the row matrix groupoid of finite order. P is a commutative matrix groupoid.

On similar lines one can define column matrix groupoid, $m \times n$ matrix groupoid $m \neq n$ or $m = n$.

All these concepts are only described by examples.

Example 2.4: Let

$$B = \{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} /$$

$$/ a_i \in G = \{Z_{16}, *, (4, 8)\}, 1 \leq i \leq 5, \times_n\}$$

be the column matrix groupoid.
Let

$$x = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 4 \\ 0 \end{bmatrix} \text{ and } y = \begin{bmatrix} 0 \\ 3 \\ 5 \\ 10 \\ 1 \end{bmatrix} \in B;$$

$$x \times_n y = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 4 \\ 0 \end{bmatrix} \times_n \begin{bmatrix} 0 \\ 3 \\ 5 \\ 10 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3*0 \\ 2*3 \\ 1*5 \\ 4*10 \\ 0*1 \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 0 \\ 0 \\ 8 \end{bmatrix} \in B.$$

Consider

$$y \times_n x = \begin{bmatrix} 0 \\ 3 \\ 5 \\ 10 \\ 1 \end{bmatrix} \times_n \begin{bmatrix} 3 \\ 2 \\ 1 \\ 4 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0*3 \\ 3*2 \\ 5*1 \\ 10*4 \\ 1*0 \end{bmatrix}$$

$$= \begin{bmatrix} 8 \\ 12 \\ 12 \\ 8 \\ 8 \end{bmatrix} \in B.$$

Clearly $x \times_n y \neq y \times_n x$.

Example 2.5: Let

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \right/$$

$a_i \in G = \{Z_{19}, (10, 10), *\}, 1 \leq i \leq 6, \times_n\}$

be the column matrix groupoid which is commutative but is non associative.

Example 2.6: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \right/$$

$a_i \in G = \{Z_{24}, *, (12, 0)\}, \times_n, 1 \leq i \leq 9\}$

be the matrix groupoid.

Let

$$x = \begin{bmatrix} 5 & 2 & 0 \\ 1 & 0 & 3 \\ 1 & 1 & 5 \end{bmatrix}$$

and

$$y = \begin{bmatrix} 0 & 3 & 6 \\ 6 & 6 & 0 \\ 2 & 0 & 0 \end{bmatrix} \in M;$$

$x \times_n y$

$$= \begin{bmatrix} 5 & 2 & 0 \\ 1 & 0 & 3 \\ 1 & 1 & 5 \end{bmatrix} \times_n \begin{bmatrix} 0 & 3 & 6 \\ 6 & 6 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 * 0 & 2 * 3 & 0 * 6 \\ 1 * 6 & 0 * 6 & 3 * 0 \\ 1 * 2 & 1 * 0 & 5 * 0 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & 0 & 0 \\ 12 & 0 & 12 \\ 12 & 12 & 12 \end{bmatrix} \in M.$$

$y \times_n x$

$$= \begin{bmatrix} 0 & 3 & 6 \\ 6 & 6 & 0 \\ 2 & 0 & 0 \end{bmatrix} \times_n \begin{bmatrix} 5 & 2 & 0 \\ 1 & 0 & 3 \\ 1 & 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 * 5 & 3 * 2 & 6 * 0 \\ 6 * 1 & 6 * 0 & 0 * 3 \\ 2 * 1 & 0 * 1 & 0 * 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 12 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\neq x \times_n y$.

Thus this matrix groupoid is non commutative and is of finite order.

Example 2.7: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \end{bmatrix} \right/$$

$a_i \in G = \{Z_{15}, *, (3, 3)\}, 1 \leq i \leq 18, \times_n\}$

be the 6×3 matrix groupoid. Clearly M is a commutative matrix groupoid which is non associative.

Now the concept of idempotents, pseudo zero divisors and pseudo nilpotent elements of order two are determined.

The notions of them are the same as in case of groupoids.

This will be illustrated by the following example.

Example 2.8: Let $M = \{(a_1, a_2, a_3, a_4) / a_i \in G = \{Z_{24}, (12, 12), *\}, 1 \leq i \leq 4, \times\}$ be the matrix groupoid.

$(0\ 0\ 0\ 0) = (g_0, g_0, g_0, g_0) \in M$ is the pseudo zero of M.

Let $x = (2\ 4\ 6\ 2)$ and $y = (8, 0, 10, 6) \in M$. Clearly

$$x \times y = (g_0, g_0, g_0, g_0) = y \times x.$$

Thus x is a pseudo zero divisor of M.
Let $x = (7, 3, 4, 3) \in M$.

$x \times x$
$= (7, 3, 4, 3) \times (7, 3, 4, 3)$
$= (7 * 7, 3 * 3, 4 * 4, 3 * 3)$
$= (g_0, g_0, g_0, g_0) \in M.$

Thus x is a pseudo nilpotent matrix of order two.

Every $x \in M$ is a pseudo nilpotent of order two. Every other pseudo zero divisor which is not a pseudo nilpotent of order two is both right and left pseudo divisor of M as M is a commutative matrix groupoid.

Example 2.9: Let

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \right/$$

$a_i \in G = \{Z_{23}, (12, 11), *\}, 1 \leq i \leq 6, \times_n\}$

be the column matrix groupoid. M has no pseudo zero divisors which are not pseudo nilpotent elements of order two. Every element $x \in M$ is a pseudo nilpotent elements of order two, that is

$$x \times_n x = \begin{bmatrix} g_0 \\ g_0 \\ g_0 \\ g_0 \\ g_0 \\ g_0 \end{bmatrix}.$$

Example 2.10: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \right/$$

$a_i \in G = \{Z_{10}, (5, 4), *\}, 1 \leq i \leq 12, \times_n\}$ be the matrix groupoid or groupoid matrix. Let

$$x = \begin{bmatrix} 2 & 5 & 2 & 0 \\ 4 & 0 & 6 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \in M$$

$x \times_n x$

$$= \begin{bmatrix} 2 & 5 & 2 & 0 \\ 4 & 0 & 6 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \times_n \begin{bmatrix} 2 & 5 & 2 & 0 \\ 4 & 0 & 6 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2*2 & 5*5 & 2*2 & 0*0 \\ 4*4 & 0*0 & 6*6 & 0*0 \\ 0*0 & 0*0 & 1*1 & 2*2 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 & 8 & 0 \\ 6 & 0 & 4 & 0 \\ 0 & 0 & 5 & 8 \end{bmatrix} \in M.$$

Let

$$y = \begin{bmatrix} 4 & 0 & 8 & 2 \\ 0 & 2 & 0 & 6 \\ 2 & 0 & 6 & 8 \end{bmatrix}$$

and

$$z = \begin{bmatrix} 5 & 0 & 5 & 0 \\ 0 & 5 & 0 & 5 \\ 5 & 5 & 5 & 5 \end{bmatrix};$$

$$y \times_n z = \begin{bmatrix} g_0 & g_0 & g_0 & g_0 \\ g_0 & g_0 & g_0 & g_0 \\ g_0 & g_0 & g_0 & g_0 \end{bmatrix}.$$

$$z \times_n y = \begin{bmatrix} 1 & 0 & 7 & 8 \\ 0 & 3 & 0 & 9 \\ 3 & 5 & 9 & 1 \end{bmatrix}.$$

Thus x is only a left pseudo zero divisor which is not a right pseudo zero divisor. The following results are true.

Theorem 2.1: Let $M = \{p \times q$ matrix with entries from $G = \{Z_n, (t, u), *\}, {}_n\}$ be the groupoid matrix. M has every element to be a pseudo nilpotent of order two if and only if G is a groupoid of the type which has pseudo nilpotents of order two (or if and only $t + u \equiv 1 \bmod n$).

Proof: As in case of groupoid G; so also in case of matrix groupoid.

Theorem 2.2: Let $M = \{p \times q$ matrix with entries from $G = \{Z_n, (t, u), *\}, {}_n\}$ be the groupoid matrix. Then M has pseudo zero divisors if and only if G has pseudo zero divisors.

Proof: As in case of groupoids. If the groupoids G has pseudo zero divisors certainly the matrix groupoids will have pseudo zero divisors.

Next the substructures like ideals and submatrix groupoids of these matrix groupoids are analysed in the following; first by illustrative examples then prove the related results.

Example 2.11: Let $M = \{(a_1, a_2, a_3) \mid a_i \in G = \{Z_{12}, (4, 6), *\}, 1 \leq i \leq 3, \times\}$ be the row matrix groupoid. $P = \{(a_1, a_2, a_3) \mid a_i \in$

$\{0, 2, 4, 6, 8, 10\}, \times\} \subseteq M$ is a matrix subgroupoid as well as the matrix ideal of M.

Example 2.12: Let

$$M = \{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} /$$

$a_i \in G = \{Z_{17}, (5, 2), *\}, 1 \leq i \leq 5, \times_n\}$
be the matrix groupoid.

$$P_1 = \{\begin{bmatrix} a_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mid a_1 \in Z_{17}, \times_n\}$$

is a matrix subgroupoid of M. Clearly P_1 is not an ideal.
Likewise

$$P_2 = \{\begin{bmatrix} 0 \\ a_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} / a_2 \in Z_{17}, \times_n\} \subseteq M$$

is again a matrix subgroupoid.

Infact there are atleast $5C_1 + 5C_2 + 5C_3 + 5C_4$ number of matrix subgroupoids of M which are not ideals of M.

$$P_9 = \{\begin{bmatrix} a_1 \\ 0 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} / a_i \in Z_{17}, \times_n, 1 \leq i \leq 4\} \subseteq M$$

is a matrix subgroupoid of M and is not an ideal.

In view of this example the following result is proved.

Theorem 2.3: Let $M = \{p \times q$ matrix with entries from $G = \{Z_p, *, (t, u)\}, t, u \in Z_p \setminus \{0\}$, p a prime, $\times_n\}$ be the matrix groupoid.

i. M has no matrix ideals.

ii. M has atleast

$(p \times q)C_1 + (p \times q)C_2 + \ldots + (p \times q)C_{p \times q - 1}$ number of matrix subgroupoids none of which are ideals.

Proof: As in case of groupoids; exploiting the fact p is a prime.

Example 2.13: Let

$$M = \{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix} /$$

$a_i \in G = \{Z_6, (0, 5), *\}, 1 \leq i \leq 8, \times_n\}$
be the matrix groupoid. M has only matrix subgroupoids and no ideals.

However if in the groupoid G the pair $(0, 5)$ is replaced by $(2, 4)$ certainly M has subgroupoids which has ideals.

This is clear from the fact G has no ideals.

Thus if the groupoid G has ideals then certainly the matrix groupoid M has ideals. However the study of the question if M the matrix groupoid has ideals does it imply G the groupoid has ideals is left for future study.

Example 2.14: Let

$$M = \{\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix} /$$

$a_i \in G = \{Z_{19}, *, (10, 3)\}; 1 \leq i \leq 6, \times_n\}$

be the matrix groupoid. G has no ideals. Clearly M has no ideals. However M has subgroupoids which are not ideals.

$$T = \{\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & a_3 \end{pmatrix} /$$

$a_i \in G = \{Z_{19}, *, (10, 3)\};$
$1 \leq i \leq 3, \times_n\} \subseteq M$

is a matrix subgroupoid of M and it is not an ideal of M.
For if

$$x = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix} \in T$$

and

$$y = \begin{pmatrix} 7 & 6 & 4 \\ 5 & 0 & 0 \end{pmatrix} \in M$$

$$x \times_n y = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix} \times_n \begin{pmatrix} 7 & 6 & 4 \\ 5 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3*7 & 0*6 & 0*4 \\ 0*5 & 2*0 & 1*0 \end{pmatrix}$$

$$= \begin{pmatrix} 13 & 18 & 12 \\ 15 & 0 & 10 \end{pmatrix} \notin T,$$

hence the claim.

Theorem 2.4: Let $M = \{p \times q$ matrices with entries from $G=\{Z_n, (t, u), *\}$, $\times_n\}$ be the matrix groupoid.

i. M has subgroupoids which are not ideals.

ii. M has $p \times q$ number of subgroupoids which are isomorphic with G.

Proof: Consider

$$P_1 = \{\begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

$/ a_1 \; Z_n, \times_n\} \subseteq M$

is a subgroupoid which is not an ideal a_1 can take any of the $p \times q$ places in the $p \times q$ matrix. Now if

$$P_i = \{\begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & a_i & \dots & 0 \end{pmatrix} / a_i \in Z_n\}$$

and it is the i^{th} place in the $p \times q$ places of the matrix. $P_i \subseteq M$ is a subgroupoid of M.
Define

$$\eta : G \to P_i \text{ by}$$

$$\eta (g) = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & g & \dots & 0 \end{pmatrix}$$

for every $g \in G$. It is easily verified $G \cong P_i$ this is true for every $P_i; 1 \;\; i \leq p \times q$. Hence M has $p \times q$ number of subgroupoids which are isomorphic to G. Hence the claim.

Next the study of idempotents in a matrix groupoid M is carried out.

First this situation is explained by some examples. Further all these matrix subgroupoids

P_i of M are not ideals of M.

Example 2.15: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{bmatrix} \right.$$

$| \, a_i \in G = \{Z_{13}, (6, 8), *\}, 1 \leq i \leq 15, \times_n\}$ be the matrix groupoid. Every $x \in M$ is an idempotent. Clearly from the fact $x * x = x$ for every $x \in G$.

Example 2.16: Let $M = \{(a_1, a_2, a_3, a_4) \,/\, a_i \in G = \{Z_{10}, (5, 6), *\}; 1 \leq i \leq 4, \times\}$ be a matrix groupoid. Infact M is an idempotent groupoid for if

$x = (7, 2, 4, 0) \in M$;

$x \times x = (7, 2, 4, 0) \times (7, 2, 4, 0)$

$= (7 * 7, 2 * 2, 4 * 4, 0 * 0)$

$= (7, 2, 4, 0)$

$= x$.

Hence the claim.

Example 2.17: Let $M = \{2 \times 5$ matrices with entries from the groupoid $\{Z_9, (4, 6), *\}, \times_n\}$ be the 2×5 matrix groupoid. Every $x \in M$ is an idempotent.

Let

$$x = \begin{pmatrix} 4 & 3 & 2 & 5 & 0 \\ 1 & 6 & 7 & 8 & 3 \end{pmatrix} \in M$$

$$x \times_n x$$

$$= \begin{pmatrix} 4 & 3 & 2 & 5 & 0 \\ 1 & 6 & 7 & 8 & 3 \end{pmatrix} \times_n \begin{pmatrix} 4 & 3 & 2 & 5 & 0 \\ 1 & 6 & 7 & 8 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 4 * 4 & 3 * 3 & 2 * 2 & 5 * 5 & 0 * 0 \\ 1 * 1 & 6 * 6 & 7 * 7 & 8 * 8 & 3 * 3 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 3 & 2 & 5 & 0 \\ 1 & 6 & 7 & 8 & 3 \end{pmatrix}$$

$= x;$

since $4 + 6 \equiv 1 \ (mod\ 9)$. M is an idempotent groupoid.

In view of this the following result is proved.

Theorem 2.5: Let $M = \{p \times q$ matrices with entries from the groupoid $G = \{Z_n, (t, u), *\}, \times_n\}$ be the matrix groupoid. M is an idempotent groupoid if and only if $t + u \equiv 1 \ (mod\ n)$.

Proof: Let $x \in M$ then $x * x = x$ if and only if $t + u \equiv 1 \ (mod\ n)$. Proof as in case of groupoids as each element in G is an idempotent and G is an idempotent groupoid. In the following section special identities satisfied by matrix groupoids and non associative semirings are described.

3. Conclusions

In this paper we introduce and study the new notion of groupoid matrices. Special identities satisfied by them is also analysed. Such study is innovative and leads to several important results in this directions.

References

1. Bruck, R.H., A survey of binary systems, Berlin Springer, Verlag, (1958).
2. Herstein, I.N., Topics in Algebra, Univ. of Chicago Press, (1969).
3. Vasantha Kandasamy, W.B., Groupoids and Smarandache Groupoids, American Research Press, Rehoboth, (2002).
4. Vasantha Kandasamy, W.B. and Florentin Smarandache, Interval Groupoids, InfoLearnQuest, Phoenix, (2010).
5. Vasantha Kandasamy, W.B. and Smarandache, F., Natural Product $\times_n$ on matrices, Zip Publishing, Ohio, (2012).