Ultra Scientist Vol. 26(2)A, 115-118 (2014).
www.ultrascientist.org

ISSN 2231-3478 (Print)
2319-8052 (Online)

Circular chromatic number of some products

R. GANAPATHY RAMAN

Assistant Professor, Department of Mathematics
Pachaiyappa's College Chennai (INDIA)

(Acceptance Date 25th May, 2014)

Abstract

Of interest in this theory is the question of determining the circular chromatic number of product graphs. There are four kinds of graph products: (i) Cartesian Product (G □ H), (ii) Direct Product (G × H), (iii) Lexicographic Product (G . H) and (iv) Strong product (G □ H).

Introduction

Star chromatic number $\chi^*(G)$ was first introduced by Vince [V] in 1988 as a generalization of the chromatic number $\chi(G)$. Bondy and Hell [BH], in 1990 modified it slightly and defined it as the infimum of certain rational numbers k/d such that G admits what is called a (k, d)-coloring. We present precise definitions in the first chapter. In 1992, Zhu[7] [Z1] took a different view point and defined a circular coloring and circular chromatic number $\chi_c(G)$. He also showed that it was equal to the star chromatic number.

The star chromatic number is at least 2. For each rational number r bigger than 2, a graph of star chromatic number r can be constructed. For any rational number r between 2 and 4, there is a planar graph with star chromatic number r.

The theory of circular chromatic number has developed rapidly. Yet, some of the straight forward and early questions are still intriguing. Vince proved in his original paper that the star chromatic number must be bigger than χ - 1 and can be at most χ. One of the questions he asked in this paper was: Which graphs have the same star chromatic number and chromatic number. But there are no known necessary and sufficient conditions yet, for a graph G to satisfy the property $\chi^*(G) = \chi(G)$. Guichard [Gu] has proved that if the chromatic number χ is not known, then it is NP-hard to determine whether or not $\chi^*(G) = \chi(G)$. Certain special classes such as graphs with superedges, wheel graphs, Mycielskian of some graphs and some Kneser graphs have the same circular chromatic number as chromatic number.

1. The Direct Product :

Definition 1.1.: The vertex set of the direct product G x H of two graphs is v(G) x v(H). Two vertices $(u_1 u_2)$, $(v_1 v_2)$ are adjacent when u_1 , $v_1 \in$ E (G) and u_2 , $v_2 \in$ E (H)

Other names for the direct product are tensor product, categorical product, Knonecker product, cardinal product, relational product, conjunction, weak direct product or just product.

Example 1.2. K_2x K_2 is the disjoint union of two edges.

Observation 1.3 E(G x H) = E(G x H) U E(G x H)

Properties 1.4 :

1. Commutative : Follows from symmetry of G and H
2. Associative: Proved as in previous cases. Here two vertices u and v are adjacent in a product of three factors if and only if all three projections into the factors consists of two distinct vertices that are adjacent in the respective factor[1,2].
3. No unit: The direct product does not have a unit. Direct product does not have a unit in $\acute{\Gamma}$ (simple graphs). If we extend the direct product to T_o (multiple edges and loops allowed)by letting two verticies (u,v), (x,y) be adjacent in G x H when ux $\in$ E(G) and vy $\in$ E(H) no matter whether u,x,v, y are distinct or not, then the graph consisting of a single vertex with a loop is a unit. As in $\acute{\Gamma}$, it is commutative, associative and distributive with respect to disjoint union.
4. Projections: p_i G_1 x….x $G_k \rightarrow G_i$ is a homomorphism, not just weak homomorphism.
5. G_i - layers are totally disconnected graphs on |G:| vertices.

Lemma 1.5 : Let (a,x), (b,y) be vertices of G x H and P a walk in G connecting a with b. Furthermore let Q be a walk from x to y in H and suppose that |E(P)| + |E(Q)| is even. Then there exists a path G x H from (a,x) to (b,y).

Proof: let P= a_o $a_{1….}a_n$ where a_o = a, a_n=b and let Q = x_0x_1 ….x_n where x_o=x and x_m=y. We may, without loss of generality assume that n<=m. Then (a_ox_o) (a_1,x_1) …(a_nx_n) (a_{n-1},x_{n+1})…(a_n,x_m) is a walk from (a,x) to (b,y) Thus, there also exists a path between these vertices.

Theorem 1.6:

Let G and H be graphs with at least one edge. Then G x H is connected if and only if both G and H are connected and at least one of them is nonbipartite, Furthermore, if both G and H are connected and bipartite, then G x H has exactly two connected components[3-5].

Proof: Let G x H be connected. Then both G and H are connected. We wish to show that at least one of the factors G and contains an odd cycle. Let xy be an arbitrary edge of H and let a be a vertex of G. Since G x H is connected, there exists a path p between (a, X) & (a,y) say p=(a_ox_o) (a_1,x_1)….(a_nx_n)=(a,y) for *i =0,1,…n* consider the pairs π_o = (d_G(a_oa_i), d_H(x_i,x_n)). Clearly π_o = (0,1) and π_n = (0,0). Note first that an entry of π_i differs by at most one from the corresponding entry in $\pi_{i-1.}$ Furthermore, for at least one pair only one entry is changed; otherwise, the parities of π_o and π_n would be the same. without loss of generality, we may assume that for some i, d_H (x_{i-1},x_n) = d_H(x_i,x_n) not equal to 0. Clearly, x_{i-1},$x_{n,}$ x_i

$\in$ E(H). But then H contains a closed walk of odd length and thus an odd cycle.

Conversely, assume that G and H are connected and the G contains an odd cycle. Consider arbitrary vertices (a,x) and (b,y) of G xH, let P be a path in G from a to b and Q be a path in H from x to y. If |E(P)| + |E(Q)| is even then (a,x) and (b,y) are connected by a path by previous lemma. Thus let |E(P)| +|E(Q)| be odd. We construct a walk P and G connecting a with b as follows. First proceed from a to a_n odd cycle of G, traverse the cycle, return back to a and continue along P. Clearly, the parity of P is different from the parity of P. Hence |E(P)| + |E(Q)| is even and we can again apply previous lemma[6,8,9].

To conclude the proof, let G and H be connected bipartite graphs. Let xy be an edge of H and let a be a vertex of G. By the above, we already know that G x H is disconnected. As before, we can show that an arbitrary vertex (c,z) of G x H is connected by a path to either (a,x) or (a,y). Thus GxH has at least two components. Since it is disconnected, it has exactly two.

Definition 1.7.

The Lexicographic product GoH of two graphs G and H is defined V(GoH) =v(G)*V(H),two vertices(u,x) and (u,y) of GoH being adjacent whenever uvεE(G) or u=v and xyεE(H). It is also known as composition or substitution.

The Lexicographic product GoH can be obtained from G by substituting a copy of Hv of H for every vertex v of G and by joining all vertices of Hv with all vertices of Hu if uvεE(G)

Theorem 1.8 : For any graph G, we have $\chi_f(G) \leq \chi_c(G) \leq \chi(G)$.

Proof: Let $\chi_f(G) = \dfrac{k}{d}$ and let c be a $(k_1 d)$ – coloring of G. For i-0, 1, …, k-1 let I_i be the set of vertices of G colored with colors {I, i+1, …., i+d-1} (colors are computed mod k). Then for any I the set Ii is an independent set of G.

Define a mapping f : s $\rightarrow$[0,1] by f $(I_i) = \dfrac{1}{d}$ for i=0, 1, …, k-1 and f(D) = 0 for any other independent set D in S. Then f is fractional coloring of G and the weight of f is $\dfrac{k}{d}$.

Therefore $\chi_f(G) \leq \dfrac{k}{d} = \chi_c(G)$.

Theorem 1.9: For any graphs G and H, we have $\chi_c(G.H) \leq \chi_c(G)\ \chi(H)$.

Proof: Let $\chi_c(G) = \dfrac{k}{d}$ and f_G be a (k, d) – coloring of G. Moreover, let $\chi(H) = n$, and f_H be an n-coloring of H. For a vertex $(a, x) \in V(G.H)$ set $f(a,x) = f_G(a) + kf_H(x)$. Then f is a mapping $v(G.H) \rightarrow \{0, 1, …., k_{n-1})$ and it is easy to see that f is a (kn,d) – coloring of G.H. Therefore $\chi_c(G.H) \leq \dfrac{kn}{d} = \chi_c(G)\ \chi(H)$.

Conclusion

We have given a very few theorems on circular chromatic number of Direct Products and lexicographic products. Feature work can also be done on circular chromatic number of other products.

References

1. Geller, r-tuple colorings of uniquely colorable graphs, *Discrete Math. 16,* 9-12 (1976).
2. Hedetniemi, Homomorphisms and graph automata, University of Michigan Technical Report 03105-44-T, (1966).
3. Wilfried Imrich, Sandi Klavžar, Product graphs, 519.17 IMR.
4. S. Klavžar, Coloring of graph products – a survey, *Discrete Math. 155,* 135-145 (1996).
5. A. Vince, Star chromatic number, J. Graph Theory *12,* 551-559 (1988).
6. X. Zhu, Star chromatic numbers and products of graphs, *J. Graph Theory 16,* 557 – 569 (1992).
7. X. Zhu, Circular chromatic number: a survey, *Discrete Math. 229,* 371-410 (2001).
8. X. Zhu, Circular chromatic number of induced subgraphs, *J. Combinatorial Theory,* Ser B, Vol. *92,* 177-181 (2004).
9. R. Ganapathy Raman and R. Sattanathan, on the circular chromatic number of deleted vertex subgraph, *Ultra Science* Vol. *20(3)M,* 709-712 (2008).