Ultra Scientist Vol. 26(2)A, 119-122 (2014).
www.ultrascientist.org

ISSN 2231-3478 (Print)
2319-8052 (Online)

Open Set and Closed Set Subspaces and the T_1 Separation Axiom

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(Acceptance Date 25th May, 2014)

Abstract

Within this paper, questions concerning open set and closed set subspaces of T_1 spaces are investigated and resolved.

Key words: subspaces, proper subspace inherited properties, T_1.

AMS Subject Classification: 54A10, 54B05, and 54D10.

1. Introduction

Given property P in a topological space, there are many questions that logically and historically arise. Included among those questions are questions concerning subspaces. Within classical topological studies, the question concerning subspaces for a property P has been "Does a space have property P if and only if every subspace has property P? Within this paper, properties for which the above statement is true are called subspace properties. In all cases for subspace properties, the proofs of the converse statement for the subspace theorem cited above are all the same and have nothing to do with the property: "Since the space is a subspace of itself and every subspace has the property, then the space has the property." As a result, proper subspace inherited properties were introduced and investigated[1] giving the properties themselves a new, major role in the investigation of subspace questions.

Definition 1.1. Let (X,T) be a space and let P be a property of topological spaces. If (X,T) has property P when every proper subspace of (X,T) has property P, then P is said to be a proper subspace inherited property[1].

Since singleton set topological spaces satisfy many topological properties, within the recent paper[1] only spaces with three or more elements were considered. Each of the subspace properties T_0, R_0, T_1, R_1, T_2, weakly Urysohn, Urysohn, regular, and T_3 proved to be proper subspace inherited properties and new characterizations for each of the properties are now known.

Theorem 1.1. A space (X,T) has property P if and only if every proper subspace of (X,T) has property P, where P can be each of the properties cited above[1].

The results above raised the question of whether or not topological properties could be further characterized using only certain types of sets within the space. With the important role of open and closed sets in the study of topology, a natural place to start such an investigation would be with open sets and closed sets, which led to new characterizations of T_0 spaces open set and closed set subspaces[2]. In this paper the T_1 separation axiom is further characterized using open set and closed set subspaces.

2. Characterizations of T_1 Spaces Using Open and Closed Set Subspaces.

As in the initial investigations cited above,[1]and[2], all spaces will have three or more elements. Also, as was the case for the investigation[2], care will be taken for proper subspaces with more than one element to ensure the resulting subspace topology is not the indiscrete topology.

Theorem 2.1. Let (X,T) be a space. Then the following are equivalent: (a) (X,T) is T_1, (b) for each open set O, (O,T_O) is T_1, (c) for each closed set C, (C,T_C) is T_1, (d) for each nonempty proper closed set C, (C,T_C) is T_1 and for each x in X, $Cl_T(\{x\})$ is not X, and (e) for each nonempty proper open set O, (O,T_O) is $T_{1,}$ for each x in X, $Cl_T(\{x\})$ is not X, and for each nonempty proper open set U in X, for each x in U, $Cl_T(\{x\})$ is a subset of U.

Proof: From the results above, (a) implies (b). Since X is open, then (X,T) is T_1 and, by the results above, (b) implies (c). Since X is closed and contains three or more elements, then (X,T) is T_1 and, by the results above, (c) implies (d).

(d) implies (e): Suppose (X,T) is not T_1. Let x be in X such that $Cl_T(\{x\})$ is not $\{x\}$. Let y be in $Cl_T(\{x\})$ not x. Then $C = Cl_T(\{x\})$ is a proper closed set in X containing both x and y and (C,T_C) is T_1. Thus $\{x\} = T_C$-closure of $\{x\} = Cl_T(\{x\})$, which contradicts y is in the $Cl_T(\{x\})$. Thus (X,T) is T_1 and, since X contains three or more element, (e) is satisfied.

(e) implies (a): Suppose (X,T) is not T_1. Let x be in X such that $Cl_T(\{x\})$ is not $\{x\}$. Let y be in $Cl_T(\{x\})$ not x. Then $U = X \backslash Cl_T(\{x\})$ is a nonempty proper open set in X. Let z be in U. Then $Cl_T(\{z\})$ is contained in U and $O = X \backslash Cl_T(\{z\})$ is a nonempty proper open set containing both x and y. Hence (O,T_O) is T_1, y is in $W = O \backslash \{x\}$, which is open in O, which is open in X, and W is open in X containing y and not x, which contradicts y is in $Cl_T(\{x\})$. Thus (X,T) is T_1.

Thus T_1 is an open set and a closed set subspace property, but not a proper open set or a proper closed set property.

3. Characterizations Using Open Set and Closed Set Subspaces of Open Set and Closed Set Subspaces:

Theorem 3.1. Let (X,T) be a space. Then the following are equivalent: (a) (X,T) is T_1, (b) for each open set O in X, for each open set W in O, (W,T_W) is T_1, (c) for each nonempty proper open set O in X, for each nonempty proper open set W in O, (W,T_W) is T_1, for each nonempty proper open set U in X, for each x in U, $Cl_T(\{x\})$ is a subset of U, and for each open set Z in X with two or more elements, for each x in Z, the T_Z-closure of {x} is not Z, (d) for each closed set C in X, for each closed set D in C, (D,T_D) is T_1, and for each closed set E in X with two or more elements, for each x in E, the T_E-closure of {x} is not E, and (e) for each nonempty proper closed set C in X, for each nonempty proper closed set D in C, (D,T_D) is T_1 and for each closed set E in X with two or more elements, for each x in E, the T_E-closure of {x} is not E.

Proof: Clearly, by Theorem 2.1, (a) implies (b).

(b) implies (c): Since X is open in X, which is open, $(X,T_X) = (X,T)$ is T_1 and $Cl_T(\{x\}) = \{x\}$ for all x in X. Thus, by Theorem 2.1, statement (c) is satisfied.

(c) implies (d): Let O be a nonempty proper open set in X. If O is a singleton set, then (O,T_O) is T_1. Thus consider the case that O contains two or more elements.

Then, for each x in O, the T_O-closure of {x} is not O. If U is a nonempty proper open set in O, then for each x in U, the T_U-closure of {x} is contained in the $Cl_T(\{x\})$, which is contained in U, and (U,T_U) is T_1. Since X is open in X and contains three or more elements, then the T_X-closure of {x} is $Cl_T(\{x\})$, which is not X. Hence, by Theorem 2.1, (O,T_O) is T_1.

Thus each nonempty proper open set in X is T_1 and, for x in X, $Cl_T(\{x\})$ is not X, and by Theorem 2.1 (X,T) is T_1, which by Theorem 2.1 implies (d).

Clearly, (d) implies (e).

(e) implies (a): Since X is closed and contains three or more elements, for each x in X, $Cl_T(\{x\})$ is not Let C be a nonempty proper closed set in X. If C is a singleton set, then (C,T_C) is T_1. Thus consider the case that C is not a singleton set.

Then for each x in C, $Cl_T(\{x\})$ equals the T_C-closure of {x} and is not C and, since each nonempty proper closed set D in C is T_1, then, by Theorem 2.1, (C,T_C) is T_1.

Hence for each nonempty proper closed set in C in X, (C,T_C) is T_1. Since for each x in X, $Cl_T(\{x\})$ is not X, by Theorem 2.1, (X,T) is T_1.

Thus additional characterizations of T_1 spaces are known giving greater understanding and insights into T_1 spaces.

References

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