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Unification of a class of trilateral generating relations with Tchebycheff polynomials

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Abstract

In this paper, a theorem in connection with the unification of a class of trilateral generating relations for certain special functions with Tchebycheffpolynomial has been proved by group theoretic method, which in turn yields a number of theorems on trilateral generating functions involving classical orthogonal polynomials (see Section 3 of this paper).

Key words: Trilateral generating relation, special functions.

AMS-2010 Subject Classification Code: 33C45,33C47,33C65.

1. Introduction

Theories in connection with the unification of bilateral or trilateral generating relations for various special functions are of greater importance in the study of problems on special functions. For previous works in this direction, one can see the works[1-7,8-15] in connection with the unification of bilateral and mixed trilateral generating relations respectively.

In the present paper, we have established a novel result in connection with the unification of trilateral generating relations for certain special functions by group theoretic method, of course when suitable continuous transformation groups can be constructed for the special functions under consideration, with Tchebycheff polynomials. The method is based on the theory of one parameter group of continuous transformations[16] by means of which any unilateral generating relation involving a special function can be transformed in to a bilateral generating relation and then into a trilateralgenerating relation with Tchebycheff polynomial by means of the relation[17]:

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(1.1) $T_n(x) = \frac{1}{2}\left[\left(x+\sqrt{x^2-1}\right)^n + \left(x-\sqrt{x^2-1}\right)^n\right].$

The detailed discussion is given below:

2. Group-Theoretic Discussion:

Let us first consider the following unilateral generating relation:

$$G(x,w) = \sum_{n=0}^{\infty} a_n\ p_n^{(\alpha+n)}(x)\ w^n, \qquad (2.1)$$

where $p_n^{(\alpha+n)}(x)$ is a special function of degree n and of parameter $(\alpha+n)$ and a_n is independent of x, w.

Replacing w by vwz and then multiplying both sides of (2.1) by y^α we get,

$$y^\alpha\, G(x,w) = \sum_{n=0}^{\infty} a_n\left(\ p_n^{(\alpha+n)}(x)y^\alpha z^n\right)(wv)^n. \qquad (2.2)$$

We now suppose that for the above special function $p_n^{(\alpha+n)}(x)$, it is possible to define a linear partial differential operator R, which generates one parameter continuous transformations group as follows:

$$R = A_1(x,y,z)\frac{\partial}{\partial x} + A_2(x,y,z)\frac{\partial}{\partial y} + A_3(x,y,z)\frac{\partial}{\partial z} + A_0(x,y,z)$$

such that

$$R\left(\ p_n^{(\alpha+n)}(x)y^\alpha z^n\right) = \rho_n\ p_{n+1}^{(\alpha+n)}(x)y^{\alpha-1}z^{n+1}, \qquad (2.3)$$

where ρ_n is function of n, α but independent of x,y,z.

Let us asume that the group generated by R as follows:

$$e^{wR}\, f(x,y,z) = \Omega(x,y,z,w) f\big(g(x,y,z,w),\ h(x,y,z,w),\ k(x,y,z,w)\big). \qquad (2.4)$$

Operating e^{wR} on both sides of (2.2), we get

$$e^{wR}\big(y^\alpha\, G(x,w)\big) = e^{wR}\left(\sum_{n=0}^{\infty} a_n\left(\ p_n^{(\alpha+n)}(x)y^\alpha z^n\right)(wv)^n\right). \qquad (2.5)$$

Now the left number of (2.5), with the help of (2.4), becomes

$$\Omega(x,y,z,w)\big(h(x,y,z,w)\big)^\alpha\, G\ \big(\ g(x,y,z,w),\ vwk(x,y,z,w)\big). \qquad (2.6)$$

The right number of (2.5), with the help of (2.3), becomes

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n \frac{w^m}{m\,!}\rho_n\ \rho_{n+1} \cdots\cdots\cdots..\rho_{n+m-1}\ p_{n+m}^{(\alpha+n)}(x)y^{\alpha-m}z^{n+m}(wv)^n. \qquad (2.7)$$

Now equating (2.6) and (2.7) and then putting $y = z = 1$, we get

$$\Omega(x,1,1,w)\big(h(x,1,1,w)\big)^\alpha\, G\ \big(\ g(x,1,1,w),\ vwk(x,1,1,w)\big)$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n \frac{w^m}{m\,!} \rho_n \ \rho_{n+1} \ldots \ldots \ldots . . \rho_{n+m-1} \ p_{n+m}^{(\alpha+n)}(x)(wv)^n$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} a_{n-m} \frac{w^n}{m\,!} \rho_{n-m} \ \rho_{n-m+1} \ldots \ldots \ldots . . \rho_{n-1} \ p_{n}^{(\alpha+n-m)}(x) \ v^{n-m}$$

$$= \sum_{n=0}^{\infty} w^n \, \sigma_n(x, v),$$

where

$$\sigma_n(x, v) = \sum_{m=0}^{n} \frac{a_m}{(n-m)\,!} \prod_{i=m}^{n-1} \rho_{\,i} \ \ p_n^{(\alpha+m)}(x) \ v^m.$$

Now to convert the above bilateral generating relation into a trilateral generating relation withTchebycheffpolynomial, we notice that

$$\sum_{n=0}^{\infty} \sigma_n(x, v) T_n(u) w^n$$

$$= \sum_{n=0}^{\infty} \sigma_n(x, v) \frac{1}{2} \left[\left(u + \sqrt{u^2 - 1} \right)^n + \left(u - \sqrt{u^2 - 1} \right)^n \right] w^n$$

$$= \frac{1}{2} \left[\sum_{n=0}^{\infty} \sigma_n(x, v) \left\{ w \left(u + \sqrt{u^2 - 1} \right) \right\}^n + \sum_{n=0}^{\infty} \sigma_n(x, v) \left\{ w \left(u - \sqrt{u^2 - 1} \right) \right\}^n \right]$$

$$= \frac{1}{2} \left[\sum_{n=0}^{\infty} \sigma_n(x, v) \, {\mu_1}^n + \sum_{n=0}^{\infty} \sigma_n(x, v) \, {\mu_2}^n \right]$$

$$= \frac{1}{2} \left[\Omega(x, 1, 1, \mu_1) \left(h(x, 1, 1, \mu_1) \right)^{\alpha} G \ \left(\ g(x, 1, 1, \mu_1), \ \ v\mu_1 k(x, 1, 1, \mu_1) \right) \right.$$

$$\left. + \, \Omega\left(x, 1, 1, \mu_2\right) \left(h\left(x, 1, 1, \mu_2\right) \right)^{\alpha} \ G \ \left(\ g\left(x, 1, 1, \mu_2\right), \ \ v\mu_2 k\left(x, 1, 1, \mu_2\right) \right) \right],$$

where

$$\mu_1 = w\left(u + \sqrt{u^2 - 1}\right) \text{and} \mu_2 = w\left(u - \sqrt{u^2 - 1}\right).$$

Thus we have prove the following theorem.

Theorem 1: If there exists a unilateral generating relation of the form:

$$G(x,w) = \sum_{n=0}^{\infty} a_n \; p_n^{(\alpha+n)}(x)\, w^n$$

then

$$\sum_{n=0}^{\infty} \sigma_n(x,v) \; T_n(u)\, w^n$$

$$= \frac{1}{2}\left[\Omega(x,1,1,\mu_1)\left(h(x,1,1,\mu_1)\right)^{\alpha} G\left(g(x,1,1,\mu_1),\; v\mu_1 k(x,1,1,\mu_1)\right)\right.$$

$$\left. + \Omega\left(x,1,1,\mu_2\right)\left(h\left(x,1,1,\mu_2\right)\right)^{\alpha} G\left(g\left(x,1,1,\mu_2\right),\; v\mu_2 k\left(x,1,1,\mu_2\right)\right)\right],$$

where

$$\sigma_n(x,v) = \sum_{m=0}^{n} \frac{a_m}{(n-m)\,!} \prod_{i=m}^{n-1} \rho_i \;\; p_n^{(\alpha+m)}(x)\, v^m,$$

$\mu_1 = w\left(u + \sqrt{u^2-1}\right)and\mu_2 = w\left(u - \sqrt{u^2-1}\right)$.

The above theorem does not seem to have appeared in the earlier works.

We now proceed to give a good number of applications of our result.[16-20]

3. Applications

Application-1: At first we take

$$p_n^{(\alpha+n)}(x) = f_n^{(\beta+n)}(x) \text{with} \alpha = \beta.$$

Then we consider the following partial differential operator R, where

$$R = x\, y^{-1} z \frac{\partial}{\partial x} - \; z \frac{\partial}{\partial y} - 2\, z^2 y^{-1} \frac{\partial}{\partial z} - x y^{-1} z$$

such that

$$R\left(\; f_n^{(\beta+n)}(x) y^{\beta} z^n \right) = -(n+1)\, f_{n+1}^{(\,\beta+n)}(x) y^{\beta-1} z^{n+1} \tag{3.1}$$

and$e^{wR} \, f(x,y,z) = \; exp(-wxy^{-1}z)$

$$\times f\left(x(1+wy^{-1}z), \frac{y}{(1+wy^{-1}z)}, \frac{z}{(1+wy^{-1}z)^2}\right). \tag{3.2}$$

Therefore comparing (2.3), (2.4) with (3.1), (3.2), we get

$$\rho_n = -(n+1), \qquad \Omega(x,y,z,w) = exp(-wxy^{-1}z),$$

$$g(x,y,z,w) = x(1+wy^{-1}z), \quad h(x,y,z,w) = \frac{y}{(1+wy^{-1}z)},$$

$$k(x,y,z,w) = \frac{z}{(1+wy^{-1}z)^2}.$$

Then, by the application of our theorem, we get the following result on trilateral generating relations with Tchebycheffpolynomials.

Theorem 2: If

$$G(x,w) = \sum_{n=0}^{\infty} a_n \ f_n^{(\beta+n)}(x)\, w^n \tag{3.3}$$

then

$$\sum_{n=0}^{\infty} \sigma_n(x,v) T_n(u) w^n$$

$$= \frac{1}{2}\Big[\exp(x\mu_1)\,(1-\mu_1)^{-\beta} G\left(x(1-\mu_1), \ \frac{\mu_1 v}{(1-\mu_1)^2}\right)$$

$$+ \exp(x\mu_2)\,(1-\mu_2)^{-\beta} G\left(x(1-\mu_2), \ \frac{\mu_2 v}{(1-\mu_2)^2}\right)\Big] \tag{3.4}$$

where

$$\sigma_n(x,v) = \sum_{k=0}^{n} a_k \binom{n}{k} \ f_n^{(\beta+k)}(x)\, v^k,$$

$$\mu_1 = w\big(u+\sqrt{u^2-1}\big)\text{and}\mu_2 = w\big(u-\sqrt{u^2-1}\big)$$

whichdose not seem to appear before.

Application-2: We now take

$$p_n^{(\alpha+n)}(x) = L_{a,\ b,\ (m+n),\ n}(x)\text{with}\alpha = m.$$

Then we consider the following partial differential operator R, where

$$R = bx\, y^{-1}z \frac{\partial}{\partial x} + \ bz \frac{\partial}{\partial y} + 2bz^2 y^{-1} \frac{\partial}{\partial z} - axy^{-1}z$$

such that

$$R\left(L_{a,\ b,\ (m+n),\ n}(x)y^m z^n\right) = (n+1)L_{a,\ b,\ (m+n),\ n+1}(x)y^{m-1}z^{n+1} \quad (3.5)$$

and

$$e^{wR}\, f(x,y,z) = \ exp\left(\frac{-awxy^{-1}z}{(1-bwy^{-1}z)}\right)$$
$$\times f\left(\frac{x}{(1-bwy^{-1}z)},\ \frac{y}{(1-bwy^{-1}z)},\ \frac{z}{(1-bwy^{-1}z)^2}\right) \quad (3.6)$$

Therefore comparing (3.5), (3.6) with (2.3), (2.4), we get

$$\rho_n = (n+1), \qquad \Omega(x,y,z,w) = exp\left(\frac{-awxy^{-1}z}{1-bwy^{-1}z}\right),$$
$$g(x,y,z,w) = \frac{x}{(1-bwy^{-1}z)},\quad h(x,y,z,w) = \ \frac{y}{(1-bwy^{-1}z)},$$
$$k(x,y,z,w) = \frac{z}{(1-bwy^{-1}z)^2}.$$

Therefore, by the application of our Theorem 1, we get the following result on trilateral generating relationwith Tchebycheff polynomials.

Theorem-3 : If

$$G(x,w) = \sum_{n=0}^{\infty} a_n L_{a,\ b,\ (m+n),\ n}(x)\, w^n \quad (3.7)$$

then

$$\sum_{n=0}^{\infty} \sigma_n(x,v)T_n(u)w^n$$
$$= \frac{1}{2}\left[\exp\left(\frac{-a\mu_1 x}{1-b\mu_1}\right)(1-b\mu_1)^{-m}\, G\left(\frac{x}{(1-b\mu_1)},\ \frac{\mu_1 v}{(1-b\mu_1)^2}\right)\right.$$
$$\left. + \exp\left(\frac{-a\mu_2 x}{1-b\mu_2}\right)(1-b\mu_2)^{-m}\, G\left(\frac{x}{(1-b\mu_2)},\ \frac{\mu_2 v}{(1-b\mu_2)^2}\right)\right], \quad (3.8)$$

where

$$\sigma_n(x,v) = \sum_{k=0}^{n} a_k \binom{n}{k} L_{a,\ b,\ (m+k),\ n}(x)\, v^k,$$
$$\mu_1 = w\left(u+\sqrt{u^2-1}\right) \text{and} \mu_2 = w\left(u-\sqrt{u^2-1}\right),$$

which does not seem to appear before.

Corollary-1: On specializing the parameters as $a = b = 1$ and $m = 1 + \alpha$ in our Theorem-3 we get the following result on Laguerre polynomials:

Theorem-4 : If

$$G(x, w) = \sum_{n=0}^{\infty} a_n L_n^{(\alpha+n)}(\mathrm{x})\, w^n \tag{3.9}$$

then

$$\sum_{n=0}^{\infty} \sigma_n(x, v) T_n(u) w^n$$

$$= \frac{1}{2}\Bigg[(1-\mu_1)^{-(1+\alpha)} \exp\left(\frac{-\mu_1 x}{1-\mu_1}\right) G\left(\frac{x}{(1-\mu_1)}, \frac{\mu_1 v}{(1-\mu_1)^2}\right)$$

$$+(1-\mu_2)^{-(1+\alpha)} \exp\left(\frac{-\mu_2 x}{1-\mu_2}\right) G\left(\frac{x}{(1-\mu_2)}, \frac{\mu_2 v}{(1-\mu_2)^2}\right)\Bigg], \tag{3.10}$$

where

$$\sigma_n(x, v) = \sum_{m=0}^{n} a_m \binom{n}{m} L_n^{(\alpha+m)}(\mathrm{x})\, v^m,$$

$$\mu_1 = w\left(u + \sqrt{u^2-1}\right) \text{and} \mu_2 = w\left(u - \sqrt{u^2-1}\right).$$

Application-3: We now take

$$p_n^{(\alpha+n)}(x) = C_n^{(\lambda+n)}(x) \text{with} \alpha = \lambda.$$

From[18], we see that

$$R = (x^2-1)\, y^{-1} z \frac{\partial}{\partial x} + 2xz \frac{\partial}{\partial y} + 3xy^{-1} z^2 \frac{\partial}{\partial z}$$

such that

$$R\left(C_n^{(\lambda+n)}(x) y^{\lambda} z^n\right) = (n+1)\, C_{n+1}^{(\lambda+n)}(x)\, y^{\lambda-1} z^{n+1} \tag{3.11}$$

and

$$e^{wR} f(x, y, z)$$

$$= f\left(\frac{xy - wz}{(wz^2 - 2wxyz + y^2)^{\frac{1}{2}}}, \frac{y^3}{(wz^2 - 2wxyz + y^2)}, \frac{zy^3}{(wz^2 - 2wxyz + y^2)^{\frac{3}{2}}}\right) \tag{3.12}$$

So by comparing (3.1),(3.12) with (2.3),(2.4), we get

$$\rho_n = (n+1),\ \ \Omega(x,y,z,w) = 1,\ \ g(x,y,z,w) = \frac{xy - wz}{(wz^2 - 2wxyz + y^2)^{\frac{1}{2}}}$$

$$h(x,y,z,w) = \frac{y^3}{(wz^2 - 2wxyz + y^2)}\ ,\ \ k(x,y,z,w) = \frac{zy^3}{(wz^2 - 2wxyz + y^2)^{\frac{3}{2}}}.$$

Then by the application of our Theorem 1, we get the following result on trilateral generating relation with Tchebycheff polynomials.

Theorem-5: If

$$G(x,w) = \sum_{n=0}^{\infty} a_n\ C_n^{(\lambda+n)}(x)\ w^n \tag{3.13}$$

Then

$$\sum_{n=0}^{\infty} \sigma_n(x,v) T_n(u) w^n$$

$$= \frac{1}{2}\left[(\mu_1 - 2\mu_1 x + 1)^{-\lambda}\, G\left(\frac{x-\mu_1}{(\mu_1 - 2\mu_1 x + 1)^{\frac{1}{2}}},\ \frac{\mu_1\, v}{(\mu_1 - 2\mu_1 x + 1)^{\frac{3}{2}}}\right)\right.$$

$$\left. +(\mu_2 - 2\mu_2 x + 1)^{-\lambda}\, G\left(\frac{x-\mu_2}{(\mu_2 - 2\mu_2 x + 1)^{\frac{1}{2}}},\ \frac{\mu_2\, v}{(\mu_2 - 2\mu_2 x + 1)^{\frac{3}{2}}}\right)\right], \tag{3.14}$$

where

$$\sigma_n(x,v) = \sum_{k=0}^{n} a_k \binom{n}{k}\ C_n^{(\lambda+k)}(x)\ v^k,$$

$\mu_1 = w\left(u + \sqrt{u^2 - 1}\right)and\mu_2 = w\left(u - \sqrt{u^2 - 1}\right)$,

which does not seem to be appear before.

Application-4: Now we take

$$p_n^{(\alpha+n)}(x) = \ \ {}_2F_1(-n,\ \beta;\ \alpha + n;\ x)$$

Then we consider the following partial differential operator R, where

$$R = x\,(1-x)y^{-1}z\frac{\partial}{\partial x} + \; z\frac{\partial}{\partial y} + 2\,z^2y^{-1}\frac{\partial}{\partial z} - \beta x y^{-1}z$$

such that

$$R(\;\;{}_2F_1(-n,\;\beta;\;\alpha+n;\;x)y^{\alpha}z^n) = (2n+\alpha)$$
$$\times\,{}_2F_1(-(n+1),\;\beta;\;\alpha+n;\;x)(x)y^{\alpha-1}z^{n+1} \qquad (3.15)$$

and

$$e^{wR}\,f(x,y,z) = \;(1-wy^{-1}z)^{\beta}\{1-wy^{-1}z(1-x)\}^{-\beta}$$
$$\times f\left(\frac{x}{1-wy^{-1}z(1-x)},\;\frac{y}{(1-wy^{-1}z)},\;\frac{z}{(1-wy^{-1}z)^2}\right) (3.16)$$

Therefore comparing (2.3), (2.4) with (3.15), (3.16), we get

$$\rho_n = (2n+\alpha), \qquad \Omega(x,y,z,w) = \;(1-wy^{-1}z)^{\beta}\{1-wy^{-1}z(1-x)\}^{-\beta},$$

$$g(x,y,z,w) = \frac{x}{1-wy^{-1}z(1-x)},\quad h(x,y,z,w) = \;\frac{y}{(1-wy^{-1}z)},$$

$$k(x,y,z,w) = \frac{z}{(1-wy^{-1}z)^2}.$$

Then by the application of our Theorem 1, we get on simplification the following result on trilateral generating relationwith Tchebycheff polynomials.

Theorem 6: If

$$G(x,w) = \sum_{n=0}^{\infty} a_n \;\;{}_2F_1(-n,\;\beta;\;\alpha+n;\;x)\,w^n \;(3.17)$$

then

$$\sum_{n=0}^{\infty}\sigma_n(x,v)T_n(u)w^n$$

$$= \frac{1}{2}\Big[(1-\mu_1)^{\beta-\alpha}\{1-\mu_1(1-x)\}^{-\beta}G\left(\frac{x}{1-\mu_1(1-x)},\qquad \frac{\mu_1 v}{(1-\mu_1)^2}\right)$$
$$+(1-\mu_2)^{\beta-\alpha}\{1-\mu_2(1-x)\}^{-\beta}G\left(\frac{x}{1-\mu_2(1-x)},\qquad \frac{\mu_2 v}{(1-\mu_2)^2}\right)\Big] \;(3.18)$$

where

$$\sigma_n(x,v) = \sum_{k=0}^{n} a_k\binom{n+\alpha+k-1}{2k+\alpha-1}\;\;{}_2F_1(-n,\;\beta;\;\alpha+k;\;x)\,v^k,$$

$\mu_1 = w\left(u + \sqrt{u^2 - 1}\right)and\mu_2 = w\left(u - \sqrt{u^2 - 1}\right)$,
which does not seem to have appeared in the earlier works.

Application 5: Now we take

$p_n^{(\alpha+n)}(x) = Y_n^{(\alpha+n)}(\mathrm{x; k})$ [19], kis a positive integer

Then from[20], we see that

$$R = x\, y^{-1} z \frac{\partial}{\partial x} + z \frac{\partial}{\partial y} + (k+1) z^2 y^{-1} \frac{\partial}{\partial z} + (1-x) y^{-1} z$$

such that

$$R\left(Y_n^{\alpha+n}(\mathrm{x; k}) y^{\alpha} z^n\right) = k(n+1) Y_{n+1}^{\alpha+n}(\mathrm{x; k}) y^{\alpha-1} z^{n+1} \tag{3.19}$$

and

$$e^{wR} f(x,y,z) = (1 - kwy^{-1}z)^{-\frac{1}{k}} \exp\left(x - \frac{x}{(1 - kwy^{-1}z)^{\frac{1}{k}}}\right)$$

$$\times f\left(\frac{x}{(1 - kwy^{-1}z)^{\frac{1}{k}}}, \frac{y}{(1 - kwy^{-1}z)^{\frac{1}{k}}}, \frac{z}{(1 - kwy^{-1}z)^{1+\frac{1}{k}}}\right) \tag{3.20}$$

Therefore comparing (2.3), (2.4) with (3.19), (3.20), we get

$$\rho_n = k(n+1), \qquad \Omega(x,y,z,w) = (1 - kwy^{-1}z)^{-\frac{1}{k}} \exp\left(x - \frac{x}{(1 - kwy^{-1}z)^{\frac{1}{k}}}\right),$$

$$g(x,y,z,w) = \frac{x}{(1 - kwy^{-1}z)^{\frac{1}{k}}}, \qquad h(x,y,z,w) = \frac{y}{(1 - kwy^{-1}z)^{\frac{1}{k}}},$$

$$k(x,y,z,w) = \frac{z}{(1 - kwy^{-1}z)^{1+\frac{1}{k}}}.$$

Then by the application of Theorem 1, we get on simplification the following result on trilateral generating relation with Tchebycheff polynomials.

Theorem7: If

$$G(x,w) = \sum_{n=0}^{\infty} a_n Y_n^{\alpha+n}(\mathrm{x; k})\, w^n \tag{3.21}$$

and

$$\sum_{n=0}^{\infty} \sigma_n(x,v) T_n(u) w^n$$

$$= \frac{1}{2}\left[(1-k\mu_1)^{\frac{-(1+\alpha)}{k}} \exp\left(x - \frac{x}{(1-k\mu_1)^{\frac{1}{k}}}\right) G\left(\frac{x}{(1-k\mu_1)^{\frac{1}{k}}}, \quad \frac{\mu_1 v}{(1-k\mu_1)^{1+\frac{1}{k}}}\right)\right.$$

$$\left. +(1-k\mu_2)^{\frac{-(1+\alpha)}{k}} \exp\left(x - \frac{x}{(1-k\mu_2)^{\frac{1}{k}}}\right) G\left(\frac{x}{(1-k\mu_2)^{\frac{1}{k}}}, \quad \frac{\mu_2 v}{(1-k\mu_2)^{1+\frac{1}{k}}}\right)\right], \quad (3.22)$$

where

$$\sigma_n(x,v) = \sum_{m=0}^{n} a_m\, k^{n-m} \binom{n}{m} Y_n^{\alpha+m}(\mathrm{x;\,k})\; v^m,$$

$$\mu_1 = w\left(u + \sqrt{u^2-1}\right) \text{and} \mu_2 = w\left(u - \sqrt{u^2-1}\right).$$

Corollary-2: On putting $k = 1$ in our Theorem 7 we get the Theorem 4.

4. Conclution

From the above discussion, it is clear that one may apply Theorem-1 in the case of other polynomials and functions existing in the field of special functions subject to the condition of construction of continuous transformation-groups for the said special function. Furthermore, the importance of the above theorems(2-7) lies in the fact that whenever one knows a unilateral generating relation of the form (3.3,3.7 etc) then the corresponding trilateral generating functions can at once be written down from (3.4,3.8etc). Thus, one can get a large number of trilateral generating functions with Tchebycheff polynomials by attributing different suitable values to a_n.

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