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A Study of the Product Summability (γ, r) $(C,1)$ to derived Fourier Series

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Abstract

The object of the present note is to extend the theorems of Sachan (E,q) (C,1) summability, qγ0 of derived Fourier series.

Key words : Cesaro Summability : (γ, *r*) summability, (γ, *r*) (*C*, 1) summability.

1. Definition & Notation

Let the Fourier series associated with the function $f(x)$ which is integrable in lebesgue sense over $(-\pi, \pi)$ and periodic with the period 2π, be

$$(1.1)\ \sum_{n=0}^{\infty} A_n(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

The derived Fourier series of (1.1) is

$$(1.2)\ \sum_{n=1}^{\infty} n\, B_n(x) = \sum_{n=1}^{\infty} n\,(b_n \cos nx - a_n \sin nx)$$

An infinite series $\sum u_n$ with the partial sums S_n is said to be summable (γ, r) $(C, 1)$ to s, if

$$\sum_{\upsilon=n}^{\infty} \binom{\upsilon}{\upsilon-n}^{(1-r)^{n+1} r^{\upsilon-n}} \sigma_\upsilon (0 \le r < 1) \to s \text{ as } n \to \infty.$$

Where σ_V stands for the $(C,1)$ transform of S_n.

We Shall use for fixed *x and s,* The following notations:

A = a positive constant, not necessarily the same at each occurrence

$\phi(t) = f(x+t) + f(x-t) - 2s$

$$g(t) = \frac{f(x+t) - f(x-t)}{4\sin t/2}$$

$p(r,t) = 1 + r^2 - 2r\cos t$

$$q(r,t) = \tan^{-1}\left\{\frac{r\sin t}{1 - r\cos t}\right\}$$

Introduction

In 1960 Ishiguro[3], in 1962 Lorch and Newmann[4] and in 1965 Forbes[1] studied asymptotic behaviour of Lebesgue constants for the Taylor Summability of Fourier Series.

It is known that the Taylor summability, which is also called the circle or (γ,r) summation is regular for $0<r<1$ and it is equivalent to ordinary convergence for $r=0$, Hardy[2] (p 218).

Studying the product summability of the derived Fourier series, Sachan[5] has proved the following theorems for (E,q) $(C,1)$ summability of the derived Fourier series.

Theorem A : If

(2.1) $G(t)=\int_0^t |g(u)|\,du = 0\left(\dfrac{t}{\log 1/t}\right),\ as\ t\to 0$

then

$$\sum_{n=1}^{\infty} n B_n(s) = 0\left[(E,q)(C,1)\right](q>0)$$

Theorem B : If

(2.2) $G(t)=\int_0^t |g(u)|\,du = 0(t),\ as\ t\to 0$

and

(2.3) $\displaystyle\int_{\pi/m}^{\eta} \frac{|g(t)-g(t+\pi/m)|}{t}\exp\left\{-\frac{nqt^2}{2\pi^2}\right\}dt = 0(1)$

as $n\to\infty$, where η is a positive constant and

$m=\dfrac{n+q+1}{q+1}$, *then* $\sum_{n=1}^{\infty} n B_n(x) = 0\left[(E,q)(C,1)\right],\ (q>0)$

So, it is natural to expect the extension of Theorems A and B to the product summability (γ,r), $(C,1)$ of the derived Fourier series under the analogous conditions. With this point of view, we prove here.

Theorem 1:

If (2.1) holds, then the derived Fourier series (1.2) is summable (γ,r), $(C,1)$ to zero at the point x.

Theorem 2:

If (2.2) holds and

(2.4) $\displaystyle\int_{\pi/m}^{\eta} \frac{|g(t)-g(t+\pi/m)|}{t}\, exp\left(-Ant^2\right)dt = o(1)$

as $n\to\infty$, for some fixed positive number $\eta<\pi$, $m=\dfrac{n+r}{1-r}$ *and* $A=\dfrac{2r}{(1+r)^2\pi^2}$ then the derived Fourier series (1.2) is summable (γ,r) $(C,1)$ to zero at the point x.

3. Preliminary Estimates and Relations:

(3.1) $q(r,t) = \tan^{-1}\dfrac{r\sin t}{1-r\cos t}$

$= \dfrac{rt}{1-r} + O(t^3)$

Where t is positive and small.

(3.2) $g(n,t) = O(n^2,t),\quad for\ 0<t<\pi/n$

$= O(n),\quad for\ 0<t<\pi/n$

(3.3) $= O(1/t),\quad for\ t>\pi/n$

For $0\le t\le\pi$ *and* $A=\dfrac{2r}{(1-r)^2\pi^2}>0$

(3.4) $\dfrac{(1-r)^{\eta}}{\{p(r,t)\}^{\eta/2}} = O\{exp(-Ant^2)\}$

(3.5) $\cos\{(n+1)(t+q(r,t)\}-\cos\left\{\dfrac{(n+1)+r}{1-r}\right\}t = O(nt^3)$

for small $t>\pi/m$.

(3.6) for $m=\dfrac{n+r}{1-r}$, finite t *and* $0<l<1$

$$\{p(r,t)\}^{-(n+1)/2} - \{p(r,t+\pi/m)\}^{-(n+1)/2}$$

$$= \frac{(n+1)\pi r(1-r)\sin(t+l\pi/m)}{(n+r)\left\{p\left(r,t+\frac{l\pi}{m}\right)\right\}^{(n+3)/2}}$$

Proof of (3.1) :

By Maclaurin's Theorem, we have

$$q(r,t) = \tan^{-1}\left(\frac{r\sin t}{1-r\cos t}\right)$$

$$= \frac{rt}{1-r} - \frac{r(1-r)t^3}{(1-r)^3 3!}$$

$$= \frac{rt}{1-r} - B(t).t^3$$

Where $B(t)$ is a bounded positive power function of t with

$$0 < B(t) < \frac{r(1-r)}{6(1-r)^3} < \frac{1}{3(1-r)^3}$$

The estimates of (3.2) can be easily be obtained by expanding sine and cosine in power of n and t, and (3.3) simply by maximising $\sin nt$ *and* $\cos nt$

Proof of (3.4) :

This estimate is due to Forbes[1] partially for $0 \le t \le \pi/2$, however the same is due to Sachan[6] completely for $0 \le t \le \pi$

Proof of (3.5) :

For $t > \pi/m$, using the relation (3.1) and expanding cosine and sine, we find

$$\cos\left\{\left((n+1)(t+q(r,t)\right)\right\} - \cos\left\{\frac{(n+1+r)}{1-r}\right\}t$$

$$= \cos\left[(n+1)\left\{t+\frac{rt}{1-r} - B(t).t^3\right\}\right] - \cos\left\{\frac{n+1+r}{1-r}\right\}t$$

$$= \cos\left[(n+1)\left\{\frac{t}{1-r} - B(t).t^3\right\}\right] - \cos\left\{\frac{n+1+r}{1-r}\right\}t$$

$$= O(nt^3)$$

Proof of (3.6) :

Since $\{p(r,t)\}^{-(n+1)/2} = (1+r^2-2r\cos t)^{-(n+1)/2}$ is continuous and differentiable in any finite interval of t, we have, by the mean value Theorem of differential calculus[6].

$$= \frac{\{p(r,t+\pi/m)\}^{-(n+1)/2} - \{p(r,t)\}^{-(n+1)/2}}{t+\frac{\pi}{m}-t}$$

$$= \frac{d}{dt}\{p(r,\theta)\}^{-(n+1)/2} \text{ Where } \theta = t+\frac{\pi l}{m},\ (0<l<1)$$

$$= -\frac{n}{2}\{p(r,\theta)\}^{-(n+3)/2}\, 2r\sin\theta$$

$$= -\frac{n}{2}\left\{p\left(r,t+\frac{l\pi}{m}\right)\right\}^{-(n+3)/2} 2r\sin\left(t+\frac{l\pi}{m}\right)$$

consequently,

$$= \{p(r,t)\}^{-(n+1)/2} - \{p(r,t+\pi/m)\}^{-(n+1)/2}$$

$$= \frac{\pi}{m} r(n+1)\left[p\left(r,t+\frac{l\pi}{m}\right)\right]^{-\frac{(n+3)}{2}} \sin\left(t+\frac{l\pi}{m}\right)$$

$$= \frac{\pi r(n+1)(1-r)}{(n+1)}\left\{p\left(r,t+\frac{l\pi}{m}\right)\right\}^{-(n+3)/2} \sin\left(t+\frac{l\pi}{m}\right)$$

$$= \frac{\pi r(1-r)\sin\left(t+\frac{l\pi}{m}\right)}{\left\{p\left(r,t+\frac{l\pi}{m}\right)\right\}^{(n+3)/2}} \text{ where } 0<l<1$$

4. Preliminary Lemmas :

Lemma 1.

If $g(t)$ be integrable (L) and δ is any positive real number less than[7] (Titchmarsh, E.C.) then the (C,1) transform σ_n of the n^{th} partial sum S_n of the derived Fourier series in given by

$$\sigma_n = \frac{2}{\pi}\int_0^{\delta} g(t)\left[\frac{2\{1-\cos(n+1)t\}}{(n+1)t^2} - \frac{\sin(n+1)t}{t}\right]dt + o(1)$$

Proof:

The n^{th} partial sum S_n of the derived Fourier series (1.2) may be written as

$$S_n = -\frac{1}{\pi}\int_0^{\pi} \psi(t)\frac{d}{dt}\left\{\frac{\sin(n+1/2)t}{2\sin t/2}\right\}dt$$

Denoting (C,1) transform of S_{n+1} by σ_n

$$\sigma_n = -\frac{1}{(n+1)\pi}\int_0^{\pi}\psi(t)\left[\sum_{k=0}^{n}\frac{d}{dt}\left\{\frac{\sin(k+1/2)t}{2\sin t/2}\right\}\right]dt$$

$$= -\frac{1}{(n+1)\pi}\int_0^{\pi}\psi(t)\frac{d}{dt}\left[\frac{1}{4\sin^2 t/2}\sum_{k=0}^{n}\{\cos kt - \cos(k+1)t\}\right]dt$$

$$= -\frac{1}{(n+1)\pi}\int_0^{\pi}\psi(t)\frac{d}{dt}\left(\frac{1-\cos(n+1)t}{\sin^2 t/2}\right)dt$$

$$= \frac{1}{\pi}\int_0^{\pi} g(t)\left[\frac{\cos\frac{t}{2}\{1-\cos(n+1)\}t}{(n+1)\sin^2 t/2} - \frac{\sin(n+1)t}{\sin t/2}\right]dt$$

$$= \frac{1}{\pi}\int_0^{\delta} g(t)\left[\frac{\cos\frac{t}{2}\{1-\cos(n+1)\}t}{(n+1)\sin^2 t/2} - \frac{\sin(n+1)t}{\sin t/2}\right]dt + o(1)$$

as $n \to \infty$, since the last interval over (δ, π) is $o(1)$ partly by presence of n in the denominator and partly by Riemann-Lebesgue Theorem.

It may also be written as

$$\sigma_n = \frac{2}{\pi}\int_0^{\delta} g(t)\left[\frac{2\{1-\cos(n+1)\}t}{(n+1)t^2} - \frac{\sin(n+1)t}{t}\right]dt + o(1)$$

Lemma 2 :

If (2.2) holds for $t \le \delta$, then

$$\int_0^{\delta} g(t)\frac{\{1-\cos(n+1)t\}}{(n+1)t^2}dt = o(1)$$

Proof:

Integral may be written as

$$\left[\int_0^{\pi/n} + \int_{\pi/n}^{\delta} g(t)\frac{\{1-\cos(n+1)t\}}{(n+1)t^2}\right]dt$$

= I + J, say

since for t > 0

$$\frac{\{1-\cos(n+1)t\}}{(n+1)t^2} = o(n)$$

We obtain by making use of (2.2)

$|I| = o(1) \quad as\ n \to \infty$

Integrating by parts and using (2.2), we get

$$|J| = 0\left(\frac{1}{n}\right)\left[0\left(\frac{1}{t}\right) + 2\int 0\left(\frac{1}{t^2}\right)dt\right]_{\pi/n}^{\delta}$$

$$= o(1), \quad as\ n \to \infty$$

It follows that

$$|I+J| \le |I| + |J| = o(1)$$

Lemma 3 :

$$\sum_{\upsilon=1}^{\infty}\binom{\upsilon}{\upsilon-n} r^{\upsilon-n}\frac{\sin(\upsilon+1)t}{t} = \frac{sin\left[(n+1)\left\{t-\tan^{-1}\left(\frac{r\sin t}{1-r\cos t}\right)\right\}\right]}{\left(1+r^2-2r\cos t\right)^{(n+1)/2}.t}$$

Proof :

$$\sum_{\upsilon=n}^{\infty}\binom{\upsilon}{\upsilon-n} r^{\upsilon-n}\frac{\sin(\upsilon+1)t}{t}$$

$$= \frac{1}{t} I_{mg} \sum_{\upsilon=n}^{\infty}\binom{\upsilon}{\upsilon-n} r^{\upsilon-n} e^{i(\upsilon+1)t}$$

$$= \frac{1}{t} I_{mg}\left[e^{i(n+1)t}\sum_{\upsilon=n}^{\infty}\binom{\upsilon}{\upsilon-n} r^{\upsilon-n} e^{i(\upsilon-n)t}\right]$$

$$= \frac{1}{t} I_{mg}\left[e^{i(n+1)t}\left(1-re^{it}\right)^{-(n+1)}\right]$$

$$= \frac{1}{t}.\frac{\sin\left[(n+1)\left\{t-\tan^{-1}\left(\frac{r\sin t}{1-r\cos t}\right)\right\}\right]}{\left(1+r^2-2r\cos t\right)^{(n+1)/2}}$$

Lemma 4 :

If $m = \frac{n+r}{1-r}$ *and* $\frac{1}{3} < \alpha < \frac{1}{2}$ and (2.2) holds. then

$$(1-r)^{(n+1)}\int_{\pi/m}^{(\pi/m)^{\alpha}} \frac{g(t)}{t}.\frac{\left[\sin\left\{(n+1)\left(t-\tan^{-1}\left(\frac{r\sin t}{1-r\cos t}\right)\right)\right\} - \sin mt\right]}{\left(1+r^2-2r\cos t\right)^{(n+1)/2}}dt$$

$= o(1)$

Proof:

By estimates (3.4, (3.5), condition (2.2) and integration by parts, we have

$$|I| \le \int_{\pi/m}^{(\pi/m)^{\alpha}} \frac{|g(t)|}{t} O\exp(-Ant^2)\ O(nt^3)dt$$

$$= 0(n)\int_{\pi/m}^{(\pi/m)^{\alpha}} |g(t)|.t^2 dt$$

$$= O(n)\left[o(t)t^2 - 2\int o(t)t\,dt\right]_{\pi/m}^{(\pi/m)^{\alpha}}$$

$$= O(n)\left[o(t^3)_{\pi/m}^{(\pi/m)^{\alpha}}\right]$$

$= o(1)$, $as\ n \to \infty$, since $m = \frac{n+r}{1-r}$

Lemma 5 :

If (2.2) holds and $0 < \alpha < \frac{1}{2}$, then

$$\left(1-r\right)^{n+1}\int_{\pi/m}^{(\pi/m)^{\alpha}}\frac{g(t+\pi/m)}{t}\left[\left\{p(r,t)\right\}^{-\frac{(n+1)}{2}}-\left\{p(r+t+\pi/m)\right\}^{-\frac{(n+1)}{2}}\sin(n+1)t\,dt\right]$$

$$= \mathrm{o}\left(1\right)$$

Proof :

Using the second mean value theorem and condition (2.2)

$$= \int_{\pi/m}^{(\pi/m)^{\alpha}}\frac{\left|g(t+\pi/m)\right|}{t}O\left[t\exp(-Ant^{2})\right]dt$$

$$= O\left[\exp\left\{-An(\pi/m)^{2\alpha}\right\}\right]\int_{\pi/m}^{(\pi/m)^{\beta}}\left|g(t+\pi/m)\right|dt$$

$$= o(1)\left[(t+\pi/m)\right]_{\pi/m}^{(\pi/m)^{\beta}} \text{ Where } 0<\alpha\le\beta<1$$

$$= \mathrm{o}\left(1\right),\quad as\ n\to\infty$$

5. Proof Of Theorem 1 :

Following lemma 1 & lemma 2, the (C,1) transform σ_n of partial sum S_n of derived Fourier series (1.2) may be written as

$$\sigma_n=-\frac{2}{\pi}\int_0^{\delta}g(t)\frac{\sin(n+1)t}{t}dt+o(1)$$

Denoting the Taylor transform of σ_n *i.e.* $\left(\gamma,r\right),(C,1)$ transform of S_n by T_n^r. We have by regularity of the Taylor method of summation,

$$T_n^r=-\frac{2}{\pi}\left(1-r\right)^{n+1}\int_0^{\delta}g(t)\sum_{\upsilon=n}^{\infty}\binom{\upsilon}{\upsilon-n}r^{\upsilon-n}\frac{\sin(n+1)t}{t}dt+o(1)$$

Using Lemma 3 and notations, we have

$$= -\frac{2}{\pi}\left(1-r\right)^{n+1}\int_0^{\delta}\frac{g(t)}{t}\frac{\sin\left[(n+1)\left\{t-\tan^{-1}\frac{r\sin t}{1-r\cos t}\right\}\right]}{(1+r^2-2r\cos t)^{(n+1)/2}}.dt+o(1)$$

$$= -\frac{2}{\pi}\left(1-r\right)^{n+1}\int_0^{\delta}\frac{g(t)}{t}\frac{\sin\left[(n+1)\left\{t-q(r,t)\right\}\right]}{\left(p(r,t)\right)^{(n+1)/2}}.dt+o(1)$$

$$= -\frac{2}{\pi}\left(1-r\right)^{n+1}\left[\int_0^{\pi/m}+\int_{\pi/m}^{(\pi/m)^{\alpha}}+\int_{(\pi/m)^{\alpha}}^{\delta}\right]\frac{g(t)}{t}\frac{\sin\left[(n+1)\left\{t-q(r,t)\right\}\right]}{\left\{p(r,t)\right\}(n+1)/2}.dt+o(1)$$

(5.1) $= -\frac{2}{\pi}\left[P+Q+R\right]+o(1)$, say where $\frac{1}{3}<\alpha<\frac{1}{2}$

We obtain by (2.1) and estimate (3.2)

$$|P| = \int_0^{\pi/m} |g(t)| O(n) dt$$

(5.2) = $o(1)$, $as\ n \to \infty$

An integration by parts and application of (2.1) gives

$$|Q| = o(1) \int_{\pi/m}^{(\pi/m)^{\alpha}} \frac{|g(t)|}{t}.dt$$

$$= o(1)\left[\left\{0(t)\frac{1}{t}\right\} + 0\left\{\frac{1}{t\log 1/t}\right\}dt\right]_{\pi/m}^{(\pi/m)^{\alpha}}$$

$$= \ o(1) + o(1)\left[-\log\log 1/t\right]_{\pi/m}^{(\pi/m)^{\alpha}}$$

$$= o(1) + 0\left[\log\frac{1}{\alpha}\right]$$

(5.3) =$o(1)$, $as\ n \to \infty$ since Where $0 < \alpha < \frac{1}{2}$

Next, by continuity part of the integral,
We have

$$|R| \ = \ \int_{(\pi/m)^{\alpha}}^{\delta} |g(t)| \exp\left\{-A(n+1)t^2\right\} dt$$

$$= \ 0\left[\frac{m^{\alpha}}{\pi^{\alpha}\exp\left\{A(n+1)\dfrac{\pi^{2\alpha}}{m^{2\alpha}}\right\}}\right]\int_{(\pi/m)^{\alpha}}^{\delta}|g(t)|dt$$

$$= \ 0\left[\frac{m^{\alpha}}{\pi^{\alpha}\exp\left\{An^{1-2\alpha}\right\}}\right]$$

(5.4) = $o(1)$, $as\ n \to \infty$ Where $0 < \alpha < \frac{1}{2}$

By virtue of (5.1), (5.2), (5.3) and (5.4)

$$T_n^{\ r} = o(1) \quad as\ n \to \infty$$

Hence the proof of theorem 1 is complete

6. Proof of Theorem 2 :

It may be noted that from the proof of theorem 1, P,R = $o(1)$ under the hypothesis (2.2) and (2.4). Applying Lemma 4, we write

$$2Q \quad = \quad \int_{\pi/m}^{(\pi/m)^{\alpha}} \frac{g(t)}{t}\frac{(1-r)^{n+1}}{\left\{p(r,t)\right\}^{(n+1)/2}}\sin mtdt + 0\,(1)$$

$$= \quad \int_{\pi/m}^{(\pi/m)^{\alpha}} \frac{g(t)}{t}\frac{(1-r)^{n+1}}{\left\{p(r,t)\right\}^{(n+1)/2}}\sin mtdt$$

$$-\int_0^{(\pi/m)^{\alpha}-\pi/m} \frac{g(t+\pi/m)}{(t+\pi/m)}\frac{(1-r)^{n+1}}{\left\{p(r,t)\right\}^{(n+1)/2}}\sin mtdt$$

$$= \quad \int_{\pi/m}^{(\pi/m)^{\alpha}} \frac{g(t)-g(t+\pi/m)}{t}\frac{(1-r)^{n+1}}{\left\{p(r,t)\right\}^{(n+1)/2}}\sin mtdt$$

$$+\int_{\pi/m}^{(\pi/m)^{\alpha}} \frac{g(t+\pi/m)}{t}(1-r)^{n+1}\left[\{p(r,t+\pi/m)\}^{-(n+1)/2}-\{p(r,t)\}^{-(n+1)/2}\right]\sin mtdt$$

$$+\int_{\pi/m}^{(\pi/m)^{\alpha}} g(t+\pi/m)\left(\frac{1}{t}-\frac{1}{t+\pi/m}\right)\frac{(1-r)^{n+1}}{\{p(r,t)\}^{(n+1)/2}}\sin mtdt$$

$$-\int_{0}^{\pi/m} \frac{g(t+\pi/m)}{t+\pi/m}\frac{(1-r)^{n+1}}{\{p(r,t+\pi/m)\}^{(n+1)/2}}\sin mtdt$$

$$+\int_{(\pi/m)^{\alpha}-(\pi/m)}^{(\pi/m)^{\alpha}} \frac{g(t+\pi/m)}{(t+\pi/m)}\frac{(1-r)^{n+1}}{\{p(r,t+\pi/m)\}^{(n+1)/2}}\sin mtdt$$

$= J_1 + J_2 + J_3 - J_4 + J_5$, say

By the hypothesis (2.3) and estimate (3.4)

$$|J_1| = o(1)\int_{\pi/m}^{(\pi/m)^{\alpha}} \frac{|g(t)-g(t+\pi/m)|}{t}\exp\{-Ant^2\}.dt$$

$= \text{o}(1), \quad as\ n \to \infty$

By (2.2) and (3.6)

$$|J_2| = \left|\int_{\pi/m}^{(\pi/m)^{\alpha}} \frac{g(t+\pi/m)}{t}\frac{(1-r)^{(n+1)}\pi r(1-r)\sin(t+\pi/m)}{(n+r)\left\{p\left(r,t+\frac{\pi l}{m}\right)^{\frac{n+3}{2}}\right\}}\sin mt\right| dt$$

$$= o(1)\int_{\pi/m}^{(\pi/m)^{\alpha}} |g(t+\pi/m)|dt$$

$= \text{o}(1), \quad as\ n \to \infty \qquad$ since, $\alpha \le \frac{1}{3}$

Integration by parts and (2.2)

$$|J_3| \le \frac{\pi}{m}\int_{\pi/m}^{(\pi/m)^{\alpha}} \frac{|g(t+\pi/m)|}{t\,(t+\pi/m)}dt$$

$$< \frac{\pi}{m}\int_{\frac{\pi}{m}}^{(\pi/m)^{\alpha}} \frac{|g(t+\pi/m)|}{t^2}dt$$

$$= o\left(\frac{1}{n}\right)\left[\frac{o(t+\pi/m)}{t^2} + 2\int \frac{o(t+\pi/m)}{t^3}dt\right]_{\pi/m}^{(\pi/m)^{\alpha}}$$

$$= o\left(\frac{1}{n}\right)\left[o\left(\frac{1}{t}\right) + 2\int o\left(\frac{1}{t^2}\right)dt\right]_{\pi/m}^{(\pi/m)^{\alpha}}$$

$= 0\ (1)$, $as\ n \to \infty$ since, $\frac{1}{3} \le \alpha < \frac{1}{2}\ and$

$$m = \frac{n+r}{1-r}$$

By change of variables and (2.2)

$$|J_4| \le \int_{\pi/m}^{2\pi/m} \frac{|g(t)|}{t}dt$$

$$= O\left(\frac{m}{\pi}\right)\int_{\pi/m}^{2\pi/m} |g(t)|dt$$

$= \mathrm{o}(1)$, $as\ n \to \infty$

and

$$|J_5| \le \int_{(\pi/m)}^{(\pi/m)^{\alpha}+\pi/m} \frac{|g(t)|}{t}.dt$$

$$< \left(\frac{m}{\pi}\right)^{\alpha} \int_{(\pi/m)^{\alpha}}^{(2\pi/m)^{\alpha}} |g(t)|dt$$

$= \mathrm{o}(1)$, $as\ n \to \infty$

Thus we show that

(5.4) = $\mathrm{o}(1)$, $as\ n \to \infty$

Which completes the proof of Theorem 2.

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