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# Average Number of Points of Inflection of a random sum of orthogonal polynomials

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**Abstract**

Let $y = \sum_{k=0}^{n} Y_k(t)\psi_k(t)$ be a random polynomial such that $(Y_0(w), Y_1(w), \ldots\ldots, Y_N(w))$ is a sequence of mutually independent normally distributed random variables with mean zero and variance one; $(\psi_0(t), \psi_1(t), \ldots\ldots, \psi_n(t))$ be a sequence of normalized Jacobi polynomials , orthogonal with respect to the interval (-1,1). It is proved that the average number of points of inflection of the random equation y=0 is asymptotic to $\sqrt{\frac{5}{7}}n$ .

*Key words :* Expected Number of Real zeros, Kac-Rice Formula, Normal Density, Jacobi Polynomial.

AMS subject classifications: 60H99, 42BXX

## 1. Introduction

Let $(Y_0(w), Y_1(w), \ldots\ldots, Y_N(w))$ be asequence of normally distributed independent random variables with mean zero and variance one. Let $(P_0^{(\alpha,\beta)}(t), P_1^{(\propto,\beta)}(t), \ldots\ldots\ldots\ldots)$ be a sequence of Jacobi polynomials. There exists a weight function w(t)= $(1-t)^{\alpha}$ $(1+t)^{\beta}$, $\alpha$, $\beta > -1$, Such that $(b_0P_0^{(\alpha,\beta)}(t), b_1P_1^{(\alpha,\beta)}(t)\ldots\ldots)$ forms an orthogonal set over the interval (-1,1) , where $b_k^{-2} = h_k = \int_{-1}^{1} w(t)P_k^{(\alpha,\beta)}$dt. Let $\psi_k(t) = b_kP_k^{(\alpha,\beta)}(t)$.

A polynomial of the form $\sum_{k=0}^{n} Y_k$(t)$\psi_k$ (t) is termed as random orthogonal polynomial. Das[2] was first to consider such a polynomial. He considered $(\psi_0(t), \psi_0(t), \ldots\ldots, \psi_n(t))$ to be a sequence of orthogonal Legendre polynomial and showed that in the interval (-1,1), all save a certain exceptional set, the expected number of real zeros of the polynomial considered by him had $\frac{n}{\sqrt{3}} + O(n)^{1/3}$ number of real zeros. Das and Bhatt[3] have calculated expected number of real zeros of random sum of Hermite polynomial. Farahamand[7] has calculated the number of level crossings of the random sum of Legendre polynomial. Wilkins[11] has calculated expected number of real zeros of a polynomial of the form $\sum_{k=0}^{n} k^p Y_k(t)\psi_k$ (t). Sahoo and Mahanti[12] have calculated average number of maxima of a random sum of orthogonal polynomials.

In this work we have calculated the average number of points of inflection of a polynomial of the form $f(t)=\sum_{k=0}^{n} Y_k(w)\, b_k P_k^{(\alpha,\beta)}(t)=\sum_{k=0}^{n} Y_k(t)\psi_k(t)$ (1.1)

We prove the following theorem

*Theorem:* If $(\psi_0(t), \psi_1(t), \ldots\ldots, \psi_n(t))$ be a sequence of normalized Jacobi polynomials, orthogonal with respect to the interval (-1,1) and $Y_0(w), Y_1(w), \ldots\ldots, Y_n(w)$ be a sequence of normally distributed independent random variables with mean zero and variance one then the average number of points of inflection of the function f(t)=0 is $\sqrt{\frac{5}{7}}\,n + 0(n^{\frac{1+\delta}{2+\delta}})$, where $\delta = \max(0, \alpha, \beta)$.

These observation, together with the fact that the totality of maxima and minima of the curve f(t)=0; and totality of points of inflection by equation f "(t) =0, lead us to conclude that

$$EP_n(f\,; a\,, b) = EN_n(f''\,;\, a\,, b) \tag{1.2}$$

where $EP_n(f\,; a\,, b)$ and E $N_n(f\,; a, b)$, respectively represent the average number of points of inflection and number of zeros of the curve f(t)=0.

It is remarkable that most of the oscillations of algebraic curve, with independent random variables as coefficients (Das[1]), occur near the neighborhood of both 1 and -1 .

we know that the same phenomenon holds good for the points of inflection of random algebraic curve . However , the scenario is distinctly different for the polynomials of the type (1 . 1). In fact just the reverse of the above observations happens as far as $EP_n(f\,; -1\,, 1)$ and $EN_n(f\,; -1\,, 1)$ are concerned. This will be evident from subsequent discussions.

Since the interior of the interval (-1,1) and the neighborhoods of the points -1 and 1 contribute ununiformly to the required estimation, we split the interval (-1,1) and subintervals (-1,-1+$\in$), (-1+$\in$,1$-\in$) and (1-$\in$,1) where, $\in = n^{-\frac{1}{2}+\delta}$. We assess the maxima in the intervals, (-1+$\in$,1$-\in$) and ,(-1+$\in$, 1$-\in$) ∪ (1$-\in$,1) in section-3 and section-4 . First we mention the formula to be used for calculation of $EM_n(f\,; a\,, b)$

## 2. *Formula for* $EM_n(f\,; a\,, b)$ :

Since evaluation $EP_n(f\,; a\,, b)$ involves estimation of zeros of $f''(t) = \sum_{k=0}^{n} y_k(w)\psi_k''(t)$, procedure applied by Kac[5] may be used to derive that

$$EP_n(f\,;a\,,b) = \frac{1}{\pi}\int_a^b \frac{[A_n(t)C_n(t) - B_n^2(t)]}{A_n(t)}\,\mathrm{dt}$$

$$= \frac{1}{\pi}\int_a^b \sqrt{\frac{C_n(t)}{A_n(t)} - \frac{B_n^2(t)}{A_n^2(t)}}\mathrm{dt}\,, \qquad (2.1)$$

Where

$A_n(t) = \sum_{k=0}^{n}[\psi_k''(t)]^2$,

$B_n(t) = \sum_{k=0}^{n}[\Psi''_k(w)\psi_k'''(t)]$,

$C_n(t) = \sum_{k=0}^{n}[\psi_k'''(t)]^2$,

Provided that $A_n(t)C_n(t) - B_n^2(t) > 0$ in (a,b) which holds good by virtue of Cauchy's inequality.

*3. Expected Number of Maxima In (-1+∈, 1-∈) :*

For convenience, we let $T_n(t) = P_n^{(\alpha,\beta)}(t)$. Let us put $\mu_n = r_n h_n^{-1} r_{n+1}^{-1}$ where $r_n$ is the coefficient of $t^n$ in $P_n^{(\alpha,\beta)}(t)$. The famous Christoffel-Darboux formula[9] for Jacobi polynomials is as follows

$$\sum_{k=0}^{k=n} h_k^{-1} P_k^{(\alpha,\beta)}(\mu) P_k^{(\alpha,\beta)}(t) = \mu_n \frac{P_{n+1}^{(\alpha,\beta)}(\mu)P_n^{(\alpha,\beta)}(t) - P_n^{(\alpha,\beta)}(\mu)P_{n+1}^{(\alpha,\beta)}(t)}{\mu - t}$$

Using the above relation and putting $\mu = \gamma + t$, we obtain

$$\sum_{k=0}^{n} h_k^{-1}T_k^2(t) + \gamma\sum_{k=0}^{n} T_k(t)h_k^{-1}T_k'(t) + \frac{\gamma^2}{2}\sum_{k=0}^{n} h_k^{-1}T_k(t)T_k''(t) + \frac{\gamma^3}{3!}\sum_{k=0}^{n} h_k^{-1}T_k(t)T_k'''(t)$$

$$+\frac{\gamma^4}{4!}\sum_{k=0}^{n} h_k^{-1}T_k(t)T_k^{iv}(t) + \frac{\gamma^5}{5!}\sum_{k=0}^{n} h_k^{-1}T_k(t)T_k^{v}(t) + \cdots$$

$$=\mu_n[T_n(t)\left\{T_{n+1}(t) + \gamma T_{n+1}'(t) + \frac{\gamma^2}{2!}T_{n+1}''(t) + \frac{\gamma^3}{3!}T_{n+1}'''(t) \ldots\right\}$$

$$-T_{n+1}\left(t\right)\left\{T_n(t) + \gamma T_n'(t) + \frac{\gamma^2}{2!}T_n''(t) + \frac{\gamma^3}{3!}T_n'''(t) + \cdots.\right\}\Big]/\gamma \qquad (3.1)$$

Comparing the coefficients of *powers of* γ on both the sides of (3.1) we have

$$\sum_{k=0}^{n} h_k^{-1}T_k(t)T_k^i(t) = \frac{\mu_n}{(i+1)}[T_{n+1}^{i+1}(t)T_n(t) - T_n^{i+1}(t)T_{n+1}(t)]\,.\,, \qquad (3.2)$$

Where $T_k^i(t)$ denotes i-th differential of $T_k(t)$ with respect to t.

Differentiating both sides of (3.1) with respect to *t* and comparing the coefficients of γ we have

$$\sum_{k=0}^{n} h_k^{-1}T_k'^2(t) + \sum_{k=0}^{n} h_k^{-1}T_k(t)T_k''(t) = \frac{\mu_n}{2}[(T_{n+1}'''(t)T_n(t) - T_{n+1}(t)T_n'''(t))$$

$$+(T_{n+1}''(t)T_n'(t) - T_{n+1}'(t)T_n''(t))]\,. \qquad (3.4)$$

Using (3.2) in (3.4) we have

$$\sum_{k=0}^{n} h_k^{-1} T_k'^2(t) = \frac{\mu_n}{6}[(T_{n+1}'''(t)T_n(t) - T_{n+1}(t)T_n'''(t)] + \frac{\mu_n}{2}[(T_{n+1}''(t)T_n'(t) - T_n''(t)T_{n+1}'(t)]. \quad (3.5)$$

Differentiating (3.1) twice with respect to and comparing the coefficients of γ on both the sides we have

$$3\sum_{k=0}^{n} h_k^{-1} T_k'(t)T_k''(t) + \sum_{k=0}^{n} h_k^{-1} T_k(t)T_k''(t) = \frac{\mu_n}{2}[(T_{n+1}^{iv}(t)T_n(t) - T_n^{iv}(t)T_{n+1}(t))$$
$$+([(T_{n+1}'''(t)T_n'(t) - T_{n+1}'(t)T_n'''(t))].$$

Using (3.2) in the above relation we have

$$\sum_{k=0}^{n} h_k^{-1}[T_k'(t)T_k''(t)] = \frac{\mu_n}{3}[(T_{n+1}'''(t)T_n'(t) - T_{n+1}'(t)T_n'''(t)] + \frac{\mu_n}{12}[(T_{n+1}^{iv}(t)T_n(t) - T_n^{iv}(t)T_{n+1}] \quad (3.6)$$

Differentiating (3.1) twice with respect to and comparing coefficient of $\gamma^2$ of both the sides we obtain

$$\sum_{k=0}^{n} h_k^{-1} T_k''^2(t) + 2\sum_{k=0}^{n} h_k^{-1} T_k'(t)T_k'''(T) + \sum_{k=0}^{n} h_k^{-1} T_k(t)T_k^{iv}(t)$$
$$=\frac{1}{3}[2(T_{n+1}^{iv}(t)T_n'(t) - T_n^{iv}(t)T_{n+1}'(t))$$
$$+ (T_{n+1}'''(t)T_n''(t) - T_n''(t)T_{n+1}''(t)) + (T_{n+1}^{v}(t)T_n(t) - T_n^{v}(t)T_{n+1}(t))]. \quad (3.7)$$

Differentiating (3.1) with respect to comparing coefficients of $\gamma^3$ we have

$$\sum_{k=0}^{n} h_k^{-1} T_k'(t)T_k'''(t) + \sum_{k=0}^{n} h_k^{-1} T_k(t)T_k'^{iv}(t)$$
$$=\frac{\mu_n}{4}[\left(T_{n+1}^{iv}(t)T_n'(t) - T_n^{iv}(t)T_{n+1}'(t)\right)+(T_{n+1}^{v}(t)T_n(t) - T_n^{v}(t)T_{n+1}(t))] \quad (3.8)$$

Using (3.2) we obtain from (3.8) that

$$\sum_{k=0}^{n} h_k^{-1} T_k'(t)T_k'''(t) = \frac{\mu_n}{4}\left(T_{n+1}^{iv}(t)T_n'(t) - T_n^{iv}(t)T_{n+1}'(t)\right)+\frac{\mu_n}{20}(T_{n+1}^{v}(t)T_n(t) - T_n^{v}(t)T_{n+1}(t)) \quad (3.9)$$

Using (3.2) and (3.7) we obtain from (3.9) that

$$\sum_{k=0}^{n} h_k^{-1} T_k''^2(t) = \frac{\mu_n}{3}[T_{n+1}'''(t)T_n''(t) - T_n'''(t)T_{n+1}''(t)] + \frac{\mu_n}{6}[T_{n+1}^{iv}(t)T_n'(t) - T_n^{iv}(t)T_{n+1}'(t)]$$
$$+\frac{\mu^n}{30}[\left(T_{n+1}^{v}(t)T_n(t) - T_n^{v}(t)T_{n+1}(t)\right)]. \quad (3.10)$$

Differentiating (3.1) twice with respect to *t* and comparing coefficient of $\gamma^3$ of both the sides we obtain

$$\sum_{k=0}^{n} h_k^{-1} T_k''(t)T_k'''(t) + 2\sum_{k=0}^{n} h_k^{-1} T_k'(t)T_k'^{v}(t) + \sum_{k=0}^{n} h_k^{-1} T T_k(t)_k^{v}(t)$$

$$=\frac{\mu_n}{4}[2\left(T^{v}_{n+1}(t)T^{'}_{n}(t)-T^{v}_{n}(t)T^{'}_{n+1}(t)\right)+(T^{iv}_{n+1}(t)T^{''}_{n}(t)-T^{iv}_{n}(t)T^{''}_{n+1}(t))$$

$$+\left(T^{vi}_{n+1}(t)T_{n}(t)-T^{vi}_{n}(t)T_{n+1}(t)\right)] \qquad (3.11)$$

Differentiating both sides of (3.1) with respect to *t* and comparing the coefficients of $\gamma^4$ and using (3.2) we have

$$\sum_{k=0}^{n} h_k^{-1}T^{'}_{k}(t)T^{iv}_{k}(t)=\frac{\mu_n}{5}[T^{v}_{n+1}(t)T^{'}_{n}(t)-T^{v}_{n}(t)T^{'}_{n+1}(t)]+\frac{\mu_n}{30}[\left(T^{vi}_{n+1}(t)T_{n}(t)-T^{vi}_{n}(t)T_{n+1}(t)\right)]. \qquad (3.12)$$

Using (3.2),( 3.11) and (3.12), we get

$$\sum_{k=0}^{n} h_k^{-1}T^{''}_{k}(t)T^{'''}_{k}(t)=\frac{\mu_n}{4}[T^{iv}_{n+1}(t)T^{''}_{n}(t)-T^{''}_{n+1}(t)T^{iv}_{n}(t)]+\frac{\mu_n}{10}[T^{v}_{n+1}(t)T^{'}_{n}(t)-T^{v}_{n}(t)T^{'}_{n+1}(t)]$$

$$+\frac{\mu_n}{60}[(T^{vi}_{n+1}(t)T_{n}(t)-T^{vi}_{n}(t)T_{n+1}(t)]. \qquad (3.13)$$

Differentiating (3.1) thrice with respect to *t* and comparing coefficient of $\gamma^3$ of both the sides we obtain

$$\sum_{k=0}^{n} h_k^{-1}T^{'''2}_{k}(t)+3\sum_{k=0}^{n} h_k^{-1}T^{''}_{k}(t)T^{iv}_{k}(t)+3\sum_{k=0}^{n} h_k^{-1}T^{'}_{k}(t)T^{v}_{k}(t)+\sum_{k=0}^{n} h_k^{-1}T_{k}(t)T^{vi}_{k}(t)$$

$$=\frac{\mu_n}{4}[3(T^{v}_{n+1}(t)T^{''}_{n}(t)-T^{v}_{n}(t)T_{n+1}(t))+3(T^{vi}_{n+1}(t)T^{'}_{n}(t)-T^{'}_{n+1}(t)T^{vi}_{n}(t))$$

$$+(T^{iv}_{n+1}(t)T^{'''}_{n}(t)-T^{iv}_{n}(t)T^{'''}_{n+1}(t))+(T^{vii}_{n+1}(t)T_{n}(t)-T^{vii}_{n}(t)T_{n+1}(t))]. \qquad (3.14)$$

Differentiating (3.1) twice with respect to *t* and comparing coefficient of $\gamma^4$ of both the sides, we obtain

$$\sum_{k=0}^{n} h_k^{-1}T^{''}_{k}(t)T^{iv}_{k}(t)+2\sum_{k=0}^{n} h_k^{-1}T^{'}_{k}(t)T^{v}_{k}(t)+\sum_{k=0}^{n} h_k^{-1}T_{k}(t)T^{vi}_{k}(t)$$

$$=\frac{\mu_n}{5}[2(T^{vi}_{n+1}(t)T^{'}_{n}(t)-T^{vi}_{n}(t)T^{'}_{n+1}(t))+(T^{v}_{n+1}(t)T^{''}_{n}(t)-T^{v}_{n}(t)T^{''}_{n+1}(t))$$

$$+\left(T^{vii}_{n+1}(t)T_{n}(t)-T^{vii}_{n}(t)T_{n+1}(t)\right)], \qquad (3.15)$$

Differentiating (3.1) with respect to *t* comparing coefficients of $\gamma^5$, we have

$$\sum_{k=0}^{n} h_k^{-1}T^{'}_{k}(t)T^{v}_{k}(t)+\sum_{k=0}^{n} h_k^{-1}T_{k}(t)T^{vi}_{k}(t)$$

$$=\frac{\mu_n}{6}[(T^{vi}_{n+1}(t)T^{'}_{n}(t)-T^{vi}_{n}(t)T^{'}_{n+1}(t))+(T^{vii}_{n+1}(t)T_{n}(t)-T^{vii}_{n}(t)T_{n+1}(t)]. \qquad (3.16)$$

Using (3.2),( 3.14),(3.15) and (3.16), we get

$$\sum_{k=0}^{n} h_k^{-1}T^{'''2}_{k}(t)=\frac{\mu_n}{4}[T^{iv}_{n+1}(t)T^{'''}_{n}(t)-T^{iv}_{n}(t)T^{'''}_{n+1}(t)]$$

$$+\frac{3\mu_n}{20}[T^{v}_{n+1}(t)T^{''}_{n}(t)-T^{''}_{n+1}(t)T^{v}_{n}(t)]+\frac{\mu_n}{20}[T^{vi}_{n+1}(t)T^{'}_{n}(t)-T^{'}_{n+1}(t)T^{vi}_{n}(t)]$$

$$+\frac{\mu_n}{140}[T^{vii}_{n+1}(t)T_{n}(t)-T^{vii}_{n}(t)T_{n+1}(t)]. \qquad (3.17)$$

From the definition of $\psi_k(t)$, we observe that

$\sum_{k=0}^{n} h_k^{-1}[T_k''(t)]^2 = \sum_{k=0}^{n}[h_k^{-1/2}T_k''(t)]^2 = \sum_{k=0}^{n}[\psi_k''(t)]^2 = A_n$

$\sum_{k=0}^{n} h_k^{-1}[T_k''(t)T_k'''(t)] = \sum_{k=0}^{n}[h_k^{-\frac{1}{2}}T_k''(t)[h_k^{-1/2}T_k'''(t)] = \sum_{k=0}^{n}\psi_k''(t)\psi_k'''(t) = B_n$

And

$\sum_{k=0}^{n} h_k^{-1}[T_k'''(t)]^2 = \sum_{k=0}^{n}[h_k^{-1/2}T_k'''(t)]\ [h_k^{-1/2}[T_k'''(t)] = \sum_{k=0}^{n}\psi_k'''(t)\psi_k'''(t) = C_n$

By the relation [17, p . 169 ], we have

$$(1-t^2)T_n''(t) = [(\alpha-\beta)+(\alpha+\beta+2)t]T_n'(t) - n(n+\alpha+\beta+1)T_n(t) \tag{3.18}$$

And

$$(1-t^2)T_{n+1}''(t) = [(\alpha-\beta)+(\alpha+\beta+2)t]T_{n+1}'(t) - (n+1)(n+\alpha+\beta+2)T_{n+1}(t) \tag{3.19}$$

Subtracting (3.18) multiplied with $T_{n+1}'(t)$ from (3.19) multiplied with $T_n'(t)$, we get

$$(1-t^2)(T_{n+1}''(t)\,T_n'(t) - T_n''(t)T_{n+1}'(t)) = n(n+1+\alpha+\beta)(T_{n+1}'(t)T_n(t) - T_n'(t)T_{n+1}(t))$$
$$-(2n+2+\alpha+\beta)T_{n+1}(t)T_n'(t). \tag{3.20}$$

Subtracting (3.18) multiplied with $T_{n+1}(t)$ from (3.20) multiplied with $T_n(t)$, we get

$$(1-t^2)(T_{n+1}''(.t)\,T_n(t) - T_n''(t)T_{n+1}(t)) = [(\alpha-\beta)+(\alpha+\beta+2)t](T_{n+1}'(t)T_n(t) - T_n'(t)T_{n+1}(t)) - (2n+2+\alpha+\beta)T_n(t)T_{n+1}(t). \tag{3.21}$$

Subtracting the derivative of (3 . 18) multiplied with $T_{n+1}(t)$ ,$T_{n+1}'(t)$, $T_{n+1}''(t)$, respectively from derivative of (3.19) multiplied with $T_n(t)$ ,$T_n'(t)$, $T_n''(t)$ , respectively, we have

$$(1-t^2)(T_{n+1}'''(t)\,T_n(t) - T_n'''(t)T_{n+1}(t)) = P_n(t)(T_{n+1}'(t)T_n(t) - T_n'(t)T_{n+1}(t)) + Q_n(t)T_n(t)T_{n+1}(t) - R_n(t)T_n^2(t). \tag{3.22}$$

where

$(1-t^2)P_n(t) = (\alpha-\beta)+(\alpha+\beta+4)t)][(\alpha-\beta)+(\alpha+\beta+2)t] + [(\alpha+\beta+2) - n(n+1+\alpha+\beta)]$

$(1-t^2)Q_n(t) = (2n+\alpha+\beta+2)[(\alpha-\beta)+(\alpha+\beta+4)t] + (n+1)[(\alpha-\beta)-(2n+\alpha+\beta+2)t]$

$(1-t^2)R_n(t) = 2(n+\alpha+1)(n+\beta+1)$

$$(1-t^2)(T_{n+1}'''(t)\,T_n'(t) - T_n'''(t)T_{n+1}'(t)) = -(2n+\alpha+\beta+2)\,(T_{n+1}'(t)T_n'(t))$$
$$+[(\alpha-\beta)+(\alpha+\beta+4)t)]\,(T_{n+1}''(t)\,T_n'(t) - T_n''(t)T_{n+1}'(t)), \tag{3.23}$$

$$(1-t^2)(T_{n+1}'''(t)\,T_n''(t) - T_n'''(t)T_{n+1}''(t))$$
$$= [n(n+1+\alpha+\beta) - (\alpha+\beta+2)](T_{n+1}''(t)T_n'(t) - T_n''(t)T_{n+1}'(t))$$
$$-(2n+2+\alpha+\beta)\,(T_{n+1}'(t)T_n''(t)), \tag{3.24}$$

Multiplying the second derivative of (3.18) with $T_{n+1}(t)$ , $T'_{n+1}(t)$ respectively and subtracting the resulting equations from second derivative of (3 .19) multiplied with $T_n(t)$ ,$T'_n(t)$ respectively, we have

$$(1-t^2)(T^{iv}_{n+1}(t)\,T_n(t)-T^{iv}_n(t)T_{n+1}(t))=-[n(n+1+\alpha+\beta)-(2\alpha+2\beta+6)](T''_{n+1}(t)T_n(t)-T''_n(t)T_{n+1}(t))+[(\alpha-\beta)+(\alpha+\beta+6)t)]\,[(T'''_{n+1}(t)\,T_n(t)-T'''_n(t)T_{n+1}(t)]$$
$$-(2n+2+\alpha+\beta)\,(T'''_n(t)T_n(t)). \qquad (3.25)$$

$$(1-t^2)(T^{iv}_{n+1}(t)\,T'_n(t)-T^{iv}_n(t)T'_{n+1}(t))=-[n(n+1+\alpha+\beta)-(2\alpha+2\beta+6)](T''_{n+1}(t)T'_n(t)-T''_n(t)T'_{n+1}(t))+[(\alpha-\beta)+(\alpha+\beta+6)t)]\,[(T'''_{n+1}(t)\,T'_n(t)-T'''_n(t)T'_{n+1}(t)]$$
$$-(2n+2+\alpha+\beta)\,(T''_{n+1}(t)T'_n(t))\,. \qquad (3.26)$$

Multiplying the third derivative of (3.18) with $T_{n+1}(t)$ and subtracting the resulting equations from third derivative of (3.12) multiplied with $T_n(t)$ we have

$$(1-t^2)(T^{v}_{n+1}(t)\,T_n(t)-T^{v}_n(t)T_{n+1}(t))=-[n(n+1+\alpha+\beta)-(3\alpha+3\beta+12)]\,(T'''_{n+1}(t)T_n(t)-T'''_n(t)T_{n+1}(t))+[(\alpha-\beta)+(\alpha+\beta+8)t)]\,[(T^{iv}_{n+1}(t)\,T_n(t)-T^{iv}_n(t)T_{n+1}(t)]$$
$$-(2n+2+\alpha+\beta)\,(T'''_{n+1}(t)T_n(t)). \qquad (3.27)$$

Similarly, we can get the following:

$$(1-t^2)(T'''_{n+1}(t)\,T''_n(t)-T''_n(t)-T'''_n(t)T''_{n+1}(t))=[n(n+1+\alpha+\beta)-(\alpha+\beta+2)](T''_{n+1}(t)T'_n(t)-T''_n(t)T'_{n+1}(t))$$
$$-(2n+\alpha+\beta+2)\,(T'_{n+1}(t)T''_n(t)). \qquad (3.28)$$

$$(1-t^2)(T^{iv}_{n+1}(t)\,T''_n(t)-T^{iv}_n(t)T''_{n+1}(t))=-(2n+\alpha+\beta+2)T''_{n+1}(t)T''_n(t)+[(\alpha-\beta)+(\alpha+\beta+6)t)]\,[(T'''_{n+1}(t)\,T''_n(t)-T'''_n(t)T''_{n+1}(t)]. \qquad (3.29)$$

$$(1-t^2)(T^{iv}_{n+1}(t)\,T'''_n(t)-T^{iv}_n(t)T'''_{n+1}(t))=n(n+1+\alpha+\beta)(T'''_{n+1}(t)T''_n(t)-T'''_n(t)T''_{n+1}(t))$$
$$-(2n+2+\alpha+\beta)\,(T''_{n+1}(t)T''_n(t)). \qquad (3.30)$$

$$(1-t^2)(T^{v}_{n+1}(t)\,T'_n(t)-T^{v}_n(t)T_{n+1}(t))=-[n(n+1+\alpha+\beta)-(3\alpha+3\beta+12)](T'''_{n+1}(t)T'_n(t)-T'''_n(t)T'_{n+1}(t))+[(\alpha-\beta)+(\alpha+\beta+8)t)]\,[(T^{iv}_{n+1}(t)\,T'_n(t)-T^{iv}_n(t)T'_{n+1}(t)]$$
$$-(2n+2+\alpha+\beta)\,T'''_{n+1}(t)T'_n(t). \qquad (3.31)$$

$$(1-t^2)(T^{v}_{n+1}(t)\,T''_n(t)-T^{v}_n(t)T''_{n+1}(t))=-[n(n+1+\alpha+\beta)-(3\alpha+3\beta+12)](T'''_{n+1}(t)T''_n(t)-T'''_n(t)T''_{n+1}(t))+[(\alpha-\beta)+(\alpha+\beta+8)t)]\,[(T^{iv}_{n+1}(t)\,T''_n(t)-T^{iv}_n(t)T''_{n+1}(t)]$$
$$-(2n+2+\alpha+\beta)\,T'''_{n+1}(t)T''_n(t) \qquad (3.32)$$

$$(1-t^2)(T^{vi}_{n+1}(t)\,T_n(t)-T^{vi}_n(t)T_{n+1}(t))=-[n(n+1+\alpha+\beta)-(4\alpha+4\beta+20)](T^{iv}_{n+1}(t)T_n(t)-T^{iv}_n(t)T_{n+1}(t))+[(\alpha-\beta)+(\alpha+\beta+10)t)]\,[(T^{v}_{n+1}(t)\,T_n(t)-T^{v}_n(t)T_{n+1}(t)]$$
$$-(2n+2+\alpha+\beta)\,T^{iv}_{n+1}(t)T_n(t). \qquad (3.33)$$

$(1-t^2)(T_{n+1}^{vi}(t)\,T_n'(t)-T_n^{vi}(t)T_{n+1}'(t))$
$=-[n(n+1+\alpha+\beta)-(4\alpha+4\beta+20)](T_{n+1}^{iv}(t)T_n'(t)-T_n^{iv}(t)T_{n+1}'(t))$
$+[(\alpha-\beta)+(\alpha+\beta+10)t)]\,[(T_{n+1}^{v}(t)\,T_n'(t)-T_n^{v}(t)T_{n+1}'(t)]$
$-(2n+2+\alpha+\beta)\,T_{n+1}^{iv}(t)T_n'(t).$ (3.34)

$(1-t^2)(T_{n+1}^{vii}(t)\,T_n(t)-T_n^{vii}(t)T_{n+1}(t))$
$=-[n(n+1+\alpha+\beta)-(5\alpha+5\beta+30)](T_{n+1}^{v}(t)T_n(t)-T_n^{v}(t)T_{n+1}(t))$
$+[(\alpha-\beta)+(\alpha+\beta+12)t)]\,[(T_{n+1}^{vi}(t)\,T_n(t)-T_n^{vi}(t)T_{n+1}(t)]$
$-(2n+2+\alpha+\beta)\,T_{n+1}^{v}(t)T_n(t)$ (3.35)

For large n, we shall use the asymptotic formula for $T_n(t)$ [2,p 33]; *i.e*

$$T_n(t)\sim\left(\frac{2^{\alpha+\beta+1}}{\pi n}\right)^{1/2}(1-t)^{-\frac{\alpha}{2}-\frac{1}{4}}(1+t)^{-\frac{\beta}{2}-\frac{1}{4}}\left[\cos\chi+O\left(\frac{1}{n\sin\theta}\right)\right]. \quad (3.36)$$

Here t=cos θ; $-1+\le t\le 1+\in$, $\in=n^{-\frac{1}{2}+\delta}$; X=$n\theta$ + Qθ+R; δ = max(0,α,β), 2Q=α+β+1, 2$r$=π(2α+1).

We first establish an asymptotic estimate for $T_{n+1}'(t)T_n(t)-T_n'(t)T_{n+1}(t)$. In view of this, we use the differentiation formula [4 , p 170 ];

$$(1-t^2)T_n'(t)=n[(\alpha-\beta)(2n+\alpha+\beta)^{-1}-t]T_n(t)+2(n+\alpha)(n+\beta)(2n+\alpha+\beta)^{-1}T_{n-1}(t), \quad (3.37)$$

Then

$$(1-t^2)\,(T_{n+1}'(t)T_n(t)-T_n'(t)T_{n+1}(t))=-[-t+(\alpha^2-\beta^2)(2n+\alpha+\beta)^{-1}(2n+2+\alpha+\beta)^{-1}]\,T_n(t)T_{n+1}(t)+2(n+1+\alpha)(n+1+\beta)\,(2n+2+\alpha+\beta)^{-1}\,T_n^2(t)-2(n+\alpha)(n+\beta)\,(2n+\alpha+\beta)^{-1}T_{n-1}(t)T_{n+1}(t). \quad (3.38)$$

Using asymptotic estimate (3.37 ), we obtain, for t∈ (−1+∈ ,1+∈),

$$(1-t^2)\,(T_{n+1}'(t)T_n(t)-T_n'(t)T_{n+1}(t))\sim 2^{\alpha+\beta-1}\pi^{-1}(1-t)^{-\alpha-1/2}(1-t)^{-\beta-1/2}\left\{1-t^2+O\left(\frac{1}{n\sin\theta}\right)\right\}. \quad (3.39)$$

Also in this range, we have

$$|T_n(t)T_{n+1}(t)|=2^{\alpha+\beta+1}(\pi n)^{-1}(1-t)^{-\alpha-1/2}(1+t)^{-\beta-1/2}\,X\left|\left\{\cos\chi+o\left(\frac{1}{n\sin\theta}\right)\right\}\left\{\cos(\chi+\theta)+O\left(\frac{1}{(n+1)\sin\theta}\right)\right\}\right|$$
$$=2^{\alpha+\beta+1}(\pi n)^{-1}(1-t)^{-\alpha-1/2}(1+t)^{-\beta-1/2}\left|\cos(\chi+\theta)+0\left(\frac{1}{n\sin\theta}\right)\right|$$
$$=0\left(\frac{(1-t)^{-\alpha-1/2}(1+t)^{-\beta-1/2}}{n}\right). \quad (3.40)$$

Similarly, we get

$$T_n^2(t)=O\left(\frac{(1-t)^{-\alpha-1/2}(1+t)^{-\beta-1/2}}{n}\right). \quad (3.41)$$

$$T_n'(t)T_{n+1}(t)=O((1-t)^{-\alpha}(1+t)^{-\beta}(1-t^2)^{-1}). \quad (3.42)$$

$$T_{n+1}'(t)T_n{}'(t)=O(n(1-t)^{-\alpha}(1+t)^{-\beta}(1-t^2)^{-2}). \quad (3.43)$$

It follows from (3.20) and (3.42) that

$$T_{n+1}''(t)T_n'(t)-T_n''(t)T_{n+1}'(t)=\frac{n^2}{(1-t^2)}(T_{n+1}'(t)T_n(t)-T_n'(t)T_{n+1}(t))+o(n(1-t)^{-\alpha}(1+t)^{-\beta}(1-t^2)^{-2}). \quad (3.44)$$

From (3.21), (3.40) we have

$$T_{n+1}''(t)T_n(t)-T_n''(t)T_{n+1}(t)=(T_{n+1}'(t)T_n(t)-T_n'(t)T_{n+1}(t))\,o((1-t)^{-1}. \quad (3.45)$$

From (3. 22), (3.40) and (3.41), we have

$$T_{n+1}'''(t)T_n(t)-T_n'''(t)T_{n+1}(t)=-\frac{n^2}{(1-t^2)}(T_{n+1}'(t)T_n(t)-T_n'(t)T_{n+1}(t))+o(n(1-t)^{-\alpha-\frac{1}{2}}(1+t)^{-\beta-\frac{1}{2}}(1-t^2)^{-1}). \quad (3.46)$$

From (3.23), (3.21) and (3.43), we have

$T'''_{n+1}(t)T'_n(t) - T'''_n(t)T'_{n+1}(t)$ = o$(n^2(1-t)^{-\alpha}$ $(1+t)^{-\beta}(1-t^2)^{-3})$. (3.47)

From (3.24), (3.20) and (3.43), we have

$T'''_{n+1}(t)T''_n(t) - T'''_n(t)T''_{n+1}(t)$

$=\frac{n^4}{(1-t^2)^2}(T'_{n+1}(t)T_n(t) - T'_n(t)T_{n+1}(t))$

$+ \text{o}(n^3(1-t)^{-\alpha}(1+t)^{-\beta}(1-t^2)^{-3})$. (3.48)

From (3.25), (3.21) and (3.22), we have

$T^{iv}_{n+1}(t)T_n(t) - T^{iv}_n(t)T_{n+1}(t)$=

$(T'_{n+1}(t)T_n(t) - T'_n(t)T_{n+1}(t))\text{o}\left(\frac{n^2}{(1-t^2)^2}\right)$. (3.49)

From (3.26), (3.20), (3.23) and (3.43) we have

$T^{iv}_{n+1}(t)T'_n(t) - T^{iv}_n(t)T'_{n+1}(t)$

$=-\frac{n^4}{(1-t^2)^2}(T'_{n+1}(t)T_n(t) - T'_n(t)T_{n+1}(t))$

$+ \text{o}(n^3(1-t^2)^{-3}(1-t)^{-\alpha}(1+t)^{-\beta})$. (3.50)

From (3.27),(3.22) and (3.26) we have

$T^{v}_{n+1}(t)T_n(t) - T^{v}_n(t)T_{n+1}(t)$

$=\frac{n^4}{(1-t^2)^2}(T'_{n+1}(t)T_n(t) - T'_n(t)T_{n+1}(t))$

$+ \text{o}(n^3(1-t)^{-\alpha}(1+t)^{-\beta}(1-t^2)^{-3})$. (3.51)

Similarly, we can get the following:

$T^{iv}_{n+1}(t)T''_n(t) - T^{iv}_n(t)T''_{n+1}(t)$

$=(T'_{n+1}(t)T_n(t) - T'_n(t)T_{n+1}(t))o\,(n^4(1-t^2)^{-3})$. (3.52)

$T^{iv}_{n+1}(t)T'''_n(t) - T^{iv}_n(t)T'''_{n+1}(t)$

$=\frac{n^4}{(1-t^2)^3}(T'_{n+1}(t)T_n(t) - T'_n(t)T_{n+1}(t))$

$+ \text{o}(n^5(1-t)^{-\alpha}(1+t)^{-\beta}(1-t^2)^{-4})$. (3.53)

$T^{v}_{n+1}(t)T'_n(t) - T^{v}_n(t)T'_{n+1}(t) = \text{o}(n^4(1-t)^{-\alpha}(1+t)^{-\beta}(1-t^2)^{-4})$. (3.54)

$T^{v}_{n+1}(t)T''_n(t) - T^{v}_n(t)T''_{n+1}(t) = \frac{n^6((T'_{n+1}(t)T_n(t) - T'_n(t)T_{n+1}(t))}{(1-t^2)^3}$

$+\text{o}(n^5(1-t)^{-\alpha}(1+t)^{-\beta}(1-t^2)^{-4})$. (3.55)

$(T^{vi}_{n+1}(t)T_n(t) - T^{vi}_n(t)T_{n+1}(t))$

$= \text{o}(n^4(1-t^2)^{-7/2}(1-t)^{-\alpha}(1+t)^{-\beta})$ (3.56)

$T^{vi}_{n+1}(t)T'_n(t) - T^{vi}_n(t)T'_{n+1}(t) = \frac{n^6((T'_{n+1}(t)T_n(t) - T'_n(t)T_{n+1}(t))}{(1-t^2)^3}$

$+\text{o}(n^5(1-t)^{-\alpha}(1+t)^{-\beta}(1-t^2)^{-4})$. (3.57)

$T^{vii}_{n+1}(t)T_n(t) - T^{vii}_n(t)T_{n+1}(t) = \frac{n^6((T'_{n+1}(t)T_n(t) - T'_n(t)T_{n+1}(t))}{(1-t^2)^3}$

$+\text{o}(n^5(1-t)^{-\alpha}(1+t)^{-\beta}(1-t^2)^{-4})$. (3.58)

From (3.10), (3.48), (3.50) and (3.51) we have

$A_n = \sum_{k=0}^{n} \text{h}_\text{k}^{-1}[\text{T}''_\text{k}(\text{t})]^2 = \sum_{k=0}^{n}[\text{h}_\text{k}^{-1/2}\text{T}''_\text{k}(\text{t})]^2 = \sum_{k=0}^{n}[\psi''_k(t)]^2$

$=\frac{\mu_n n^4}{5(1-t^2)^2}(T'_{n+1}(t)T_n(t) - T'_n(t)T_{n+1}(t)\left[1 + o\left(\frac{1}{n(1-t^2)^{\frac{1}{2}}}\right)\right]$. (3.59)

From (3.13), (3.52), (3.54) and (3.56), we have

$B_n(t) = \sum_{k=0}^{n} \psi''_k(t)\,\psi'''_k(t)$

$=\mu_n\ \text{o}\left(n^4(1-t)^{-\alpha}(1+t)^{-\beta}(1-t^2)^{-4}\right)$. (3.60)

From (3.17), (3.53), (3.55), (3.57) and (3.58), we have

$C_n(t) = \sum_{k=0}^{n}[\psi'''_k(t)]^2$

$=\mu_n[\frac{n^6}{4(1-t^2)^2}(T'_{n+1}(t)T_n(t) - T'_n(t)T_{n+1}(t))$

$-\frac{3n^6}{20(1-t^2)^3}(T'_{n+1}(t)T_n(t) - T'_n(t)T_{n+1}(t))$

$+\frac{n^6}{20(1-t^2)^3}\left(T'_{n+1}(t)T_n(t) - T'_n(t)T_{n+1}(t)\right)$

$-\frac{n^6}{140(1-t^2)^2}\left(T'_{n+1}(t)T_n(t) - T'_n(t)T_{n+1}(t)\right)$

$+ \text{o}(n^5(1-t)^{-\alpha}(1+t)^{-\beta}(1-t^2)^{-4})]$

$=\frac{\mu_n n^6(T'_{n+1}(t)T_n(t) - T'_n(t)T_{n+1}(t))}{7(1-t^2)^2}\left[1 + o\left(\frac{1}{n(1-t^2)^{\frac{1}{2}}}\right)\right]$ (3.61)

From (3.59) and (3.61), we have

$\frac{C_n(t)}{A_n(t)} \sim \sqrt{\frac{5}{7}}\frac{n^2}{(1-t^2)}$.

From (3.59) and (3.60), we have

$\frac{B_n(t)}{A_n(t)} = 0\left((1-t^2)^{-\frac{5}{2}}\right)$,

So that

$$\sqrt{\frac{C_n(t)}{A_n(t)}-(\frac{B_n(t)}{A_n(t)})^2}=\sqrt{\frac{5}{7}}\frac{n^2}{(1-t^2)^{1/2}}\left[1+0\left(n^{-\frac{2\delta}{2+\delta}}\right)\right].$$

From (2.1), we obtain

$$EP_n(f;\ -1+\in,1+\in)=\frac{1}{\pi}\int_{-1+\in}^{1-\in}\sqrt{\frac{5}{7}}\frac{n}{(1-t^2)^{1/2}}\left[1+0\left(n^{-\frac{2\delta}{2+\delta}}\right)\right] \quad (3.62)$$

$$\sim\sqrt{\frac{5}{7}}n$$

*4. Number of Point of Inflection in the Ranges (1-∈, 1) and (-1 , -1+∈) :*

In this section we obtain that ,in comparison to the interval $(-1+\in,1-\in)$, equation (1.1) has negligi ble number of points of inflection in the intervals (1-∈,1) and (-1, -1+∈). In fact, it is sufficient to prove that the number of zeros of $\sum_{k=0}^{n} y_k(w)\psi_k''(t)=0$ is very less in these two intervals.

We put

$$\emptyset(z)=\emptyset(y(w),z)=\sum_{k=0}^{n} y_k(w)\psi_k''(t), \quad (4.1)$$

where y(w) denotes the random vector $(Y_0(w),Y_1(w),\dots\dots\dots Y_n(w))$.

Now $\emptyset(y(w),1)=\sum_{k=0}^{n} y_k(w)\psi_k''(1)$, is a random variable with distribution function $\frac{1}{\sqrt{2\pi\Lambda^2}}\int_{-\infty}^{t} exp\left(-\frac{1}{2}\frac{u^2}{\Lambda^2}\right)du$, here $\Lambda^2=\sum_{k=0}^{n}\psi_k''^2(1)>0.$

Hence

$$P(|\emptyset(1)|<e^{-2ne})=\left(\frac{2}{\pi\Lambda^2}\right)^{1/2}\int_0^{-2ne} exp\left(-1/2\frac{u^2}{\Lambda^2}\right)du\ \le e^{-ne}. \quad (4.2)$$

If $I_n=\max_{0\le k\le n}|y_k(w)|$, then $|\emptyset(1+\in e^{i\theta})|\le I_n\{\sum_{k=0}^{n}|\psi_k''(1+\in e^{i\theta})|\}$.

It is easy to see that $P(I_n\le n)>1-e^{-n^2/2}$.

Let $M_n=\max_{0\le k\le n}|\psi_k''(1+\in e^{i\theta})|$.

Then $P(\phi(1+\in e^{i\theta})\le n^2M_n)\ge 1-e^{-n^2/2}$.(4.3)

By the Schalfi representation of Jacobi polynomials [17, p 172], we have

$$P_k^{(\alpha,\beta)}(\xi)=\frac{1}{2\pi i}\oint_{\xi+}\left(\frac{t^2-1}{2}\right)^k\left(\frac{1-t}{1-\xi}\right)^\alpha\left(\frac{1-t}{1+\xi}\right)^\beta\frac{dt}{(t-\xi)^{k+1}}, \quad (4.4)$$

For simple closed contour (ξ+) around the point (ξ) as center in the positive sense, with radius $|\xi^2-1|^{1/2}$, so taken that it has the points -1 or 1, neither in the interior nor on the boundary.

Putting $t=\xi+(\xi^2-1)^{1/2}e^{i\theta}$, after differen-tiating (4.4) with respect to , we have

$$T_n'(\xi)=\frac{1}{2\pi}\left[\alpha\int_0^{2\pi}\left\{\left(\left(\xi+i\sqrt{1-\xi^2}cos\phi\right)\right)^{n+1}\left(1-i\sqrt{\frac{1-\xi}{1+\xi}}e^{i\phi}\right)^\alpha\left(1+i\sqrt{\frac{1-\xi}{1+\xi}}e^{i\phi}\right)^\beta\left(\frac{1}{1-\xi}\right)\right\}d\phi-\right.$$

$$\beta\int_0^{2\pi}\left\{\left(\xi+i\sqrt{1-\xi^2}cos\phi\right)^{n+1}\left(1-i\sqrt{\frac{1-\xi}{1+\xi}}e^{i\phi}\right)^\alpha\left(1+i\sqrt{\frac{1-\xi}{1+\xi}}e^{i\phi}\right)^\beta\left(\frac{1}{1+\xi}\right)\right\}d\phi+(n+1)\int_0^{2\pi}\left\{\left(\xi+\right.\right.$$

$$\left.\left.i\sqrt{1-\xi^2}cos\phi\right)^{n+1}\left(1-i\sqrt{\frac{1-\xi}{1+\xi}}e^{i\phi}\right)^\alpha\left(1+\sqrt{\frac{1-\xi}{1+\xi}}e^{i\phi}\right)^\beta\frac{1}{(\xi^2-1)^{1/2}e^{i\theta}}\right\}d\theta\ \right].$$

Hence

$$|T_n'(1+\in e^{i\theta})| < \frac{c_1(n+1)}{\in}(1+2\in)^n(1+2/\in)^{\alpha}(1+2\in)^{\beta}, \quad (4.5)$$

where $c_1$ is constant. For Jacobi Polynomials, $h_n$ is determined by

$$n!\,(1+\alpha+\beta+2n)\Gamma\,(1+\alpha+\beta+n) = 2^{1+\alpha+\beta}\Gamma(1+\alpha+n)\Gamma(1+\beta+n)$$

hence $M_n = \frac{max}{n}|\psi_n''(1+\in e^{i\theta})| = \frac{max}{n}\left|h_n^{-\frac{1}{2}}T_n'(t)\right|$

$$< C_2 n^A exp[4n\in] = C_2 n^A exp\left[4n^{\frac{1+\delta}{2+\delta}}\right], \quad (4.6)$$

where $c_2$ and A are constants.

From (4.2), (4.3) and (4.6), we conclude that,

$$\left|\frac{\emptyset(1+\in e^{i\theta})}{\emptyset(1)}\right| \le C_2 n^{2+A} exp\left[6n^{\frac{1+\delta}{2+\delta}}\right]. \quad (4.7)$$

with probability at least 1-exp $(-\frac{n^2}{2})$-exp (-n∈)> $1-\frac{1}{n}$,

If n(∈) denote the number of zeros of ϕ(z) in |z-1|<∈, then by application of Jensen's theorem we find that

$$n(\in) < \frac{1}{2\pi log2}\int_0^{2\pi} log\left|\frac{\emptyset(1+\in e^{i\theta})}{\emptyset(1)}\right| d\theta = 0(n^{\frac{1+\delta}{2+\delta}}), \quad (4.8)$$

with probability at least 1-1/n.

Thus the number of zeros of ϕ(z)=0; *i. e* the number of points of inflection of (4.1) in the region of |z-1|<∈ and hence that of (1.1) in the interval (1-∈,1), is $0(n^{\frac{1+\delta}{2+\delta}})$. Similarly we can prove that the equation (1.1) has $0(n^{\frac{1+\delta}{2+\delta}})$ points of inflection in (-1 , -1+∈).

Bringing together the conclusions drawn[3,4], we prove that

$$EP_n(f;-1,1) = \sqrt{\frac{5}{7}}\,n + o\left(n^{\frac{1+\delta}{2+\delta}}\right) \sim \sqrt{\frac{5}{7}}\,n$$

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