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On mixed trilateral generating relations for biorthogonal polynomials suggested by the Laguerre polynomials

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Abstract

In this note, we have obtained some novel results on mixed trilateral generating relations involving the polynomials, $Y_{n+r}^{a-nk}(x;k)$ a modified form ofKonhauser biorthogonal polynomials, $Y_n^a(x;k)$ by group theoretic method. As special cases, we have obtained the corresponding results on generalized Laguerre polynomials. Some applications of our results are also discussed.

Key words: Laguerre polynomials, biorthogonal polynomials, mixed trilateral generating functions.

AMS-2010 Subject Classification Code: 33C 45, 33C 47.

1. Introduction

The polynomial sets { $Y_n^a(x;k)$ } and { $Z_n^\alpha(x;k)$ }, discussed by J.D.E. Konhauser[1] are biorthogonal with respect to the weight function $x^\alpha e^{-x}$ over the interval $(0,\infty)$, $\alpha > 1$, k is a positive integer. For $k = 1$, these polynomials reduce to the generalized Laguerre polynomials, $L_n^\alpha(x)$. An explicit expression for the polynomials, $Y_n^\alpha(x;k)$ was given by Carlitz[2] in the following form:

$$Y_n^\alpha(x;k) = \frac{1}{n!}\sum_{i=0}^{n}\frac{x^i}{i!}\sum_{j=0}^{i}(-1)^j\binom{i}{j}\left(\frac{j+\alpha+1}{k}\right)_n,$$

where $(a)_n$ is the pochhammersymbol defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & if\ n=0,\ a \neq 0, \\ a(a+1)\ldots(a+n-1), & \forall\, n \in \{1,2,3\ldots\} \end{cases}$$

In a recent paper[7], the present authors have proved the following theorems on bilateral generating relations involving the polynomials, $Y_{n+r}^{\alpha\text{-}nk}(x;k)$, a modified form of Konhauser biorthogonal polynomials, $Y_n^{\alpha}(x;k)$.

Theorem 1: If

$$G(x,w) = \sum_{n=0}^{\infty} a_n Y_{n+r}^{\alpha\text{-}nk}(x;k) w^n \quad (1.1)$$

then

$$(1=kt)^{\frac{(1+\alpha-k)}{k}} \exp\left\{x - x(1+kt)^{\frac{1}{k}}\right\} G\left(x(1+kt)^{\frac{1}{k}}, \frac{vt}{1+kt}\right)$$

$$= \sum_{n=0}^{\infty} \sigma_n(v) Y_{n+r}^{\alpha\text{-}nk}(\mathrm{x};\mathrm{k}) t^n, \quad (1.2)$$

where

$$\sigma_n(v) = \sum_{p=0}^{\infty} a_p k^{n\text{-}p} \binom{n+r}{p+r} v^p.$$

Theorem 2 If

$$G(x,w) = \sum_{n=0}^{\infty} a_n Y_{n+r}^{\alpha}(\mathrm{x};\mathrm{k}) w^n \quad (1.3)$$

then

$$(1=kt)^{\frac{(1+\alpha-k)}{k}} \exp\left\{x - x(1+kt)^{\frac{1}{k}}\right\} G\left(x(1+kt)^{\frac{1}{k}}, vt\right)$$

$$= \sum_{n=0}^{\infty} \sigma_n(x,v) t^n, \quad (1.4)$$

where

$$\sigma_n(x,v) = \sum_{p=0}^{\infty} a_p k^{n\text{-}p} \binom{n+r}{p+r} Y_{n+r}^{\alpha\text{-}nk+pk}(\mathrm{x};\mathrm{k}) v^p.$$

The object of the present paper is to generalise the above bilateral generating relationsinto mixed trilateral generating relations by the group-theoretic method. As special cases, we obtain the corresponding results on Laguerre polynomials, $L_n^{(\alpha)}(x)$. The main results of our investigation are stated in the form of the following theorems:

Theorem 3 If

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n Y_{n+r}^{\alpha-nk}(\mathrm{x};\mathrm{k}) g_n(u) w^n, \quad (1.5)$$

where $g_n(u)$ is an arbitrary polynomial of degree n, then

$$(1=kt)^{\frac{(1+\alpha-k)}{k}} \exp\left\{x - x(1+kt)^{\frac{1}{k}}\right\} G\left(x(1+kt)^{\frac{1}{k}}, u, \frac{vt}{1+kt}\right)$$

$$= \sum_{n=0}^{\infty} \sigma_n(v,u) Y_{n+r}^{\alpha-nk}(\mathrm{x};\mathrm{k}) t^n, \quad (1.6)$$

where

$$\sigma_n(u,v) = \sum_{p=0}^{n} a_p k^{n\text{-}p} \binom{n+r}{p+r} g_p(u) v^p.$$

Theorem 4 If

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n Y_{n+r}^{\alpha}(\mathrm{x};\mathrm{k}) g_n(u) w^n, \quad (1.7)$$

where $g_n(u)$ is an arbitrary polynomial of degree then

$$1=kt)^{\frac{(1+\alpha-k)}{k}} \exp\left\{x - x(1+kt)^{\frac{1}{k}}\right\} G\left(x(1+kt)^{\frac{1}{k}}, u, vt\right)$$

$$= \sum_{n=0}^{\infty} \sigma_n(x,u,v) t^n, \quad (1.8)$$

where

$$\sigma_n(x,u,v) = \sum_{p=0}^{n} a_p k^{n-p} \binom{n+r}{p+r} Y_{n+r}^{\alpha-nk+pk}(x;k) g_p(u) v^p.$$

2. Proof of theorem 3 :

At first, we consider the following linear partial differential operator[7]:

$$R = xy\frac{\partial}{\partial x} - ky^2\frac{\partial}{\partial y} - (x+k-\alpha-1)y$$

such that

$$R(Y_{n+r}^{\alpha-nk}(x;k)y^n) = k(n+r+1)Y_{n+r+1}^{\alpha-nk-k}(x;k)y^{n+1}. \tag{2.1}$$

The extended form of the group generated by R is given by

$$e^{wR} f(x,y) = (1+kwy)^{\frac{1+\alpha-k}{k}} \exp\left[x - x(1+kwy)^{\frac{1}{k}}\right]$$

$$\times f\left(x(1+kwy)^{\frac{1}{k}}, \frac{y}{1+kwy}\right), \tag{2.2}$$

where $f(x,y)$ is an arbitrary function and w is an arbitrary constant.

Let us consider the generating relation of the form:

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n Y_n^{\alpha-nk}(x;k) g_n(u) w^n. \tag{2.3}$$

Replacingbyin the both sides of (2.3) we have

$$G(x,u,wvy) = \sum_{n=0}^{\infty} a_n (Y_{n+r}^{\alpha-nk}(x;k) g_n(u) y^n)(wv)^n. \tag{2.4}$$

Operating e^{wR} on both sides of (2.4), we get

$$e^{wR}\left[G(x,u,wvy)\right] = e^{wR}\left[\sum_{n=0}^{\infty} a_n (Y_{n+r}^{\alpha-nk}(x;k) g_n(u) y^n)(wv)^n\right]. \tag{2.5}$$

Now the left member of (2.5), with the help of (2.2), reduces to

$$(1+kwy)^{\frac{1+\alpha-k}{k}} \exp\left[x - x(1+kwy)^{\frac{1}{k}}\right]$$

$$G\left(x(1+kwy)^{\frac{1}{k}}, u, \frac{wvy}{1+kwy}\right). \tag{2.6}$$

The right member of (2.5), with the help of (2.1), becomes

$$= \sum_{n=0}^{\infty}\sum_{p=0}^{n} a_{n-p} \frac{w^p}{p!} k^p (n+r-p+1)_p$$

$$Y_n^{\alpha-nk}(x;k) g_{n-p}(u) y^n (wv)^{n-p}. \tag{2.7}$$

Now equating (2.6) and (2.7) and then substituting $wy=t$, we get

$$(1+kwy)^{\frac{1+\alpha-k}{k}} \exp\left[x - x(1+kt)^{\frac{1}{k}}\right] G\left(x(1+kt)^{\frac{1}{k}}, u, \frac{vt}{1+kt}\right)$$

$$= \sum_{n=0}^{\infty} Y_n^{\alpha-nk}(x;k)\sigma_n(u,v)t^n, \tag{2.8}$$

where

$$\sigma_n(u,v) = \sum_{p=0}^{n} a_p k^{n-p}\binom{n+r}{p+r} g_p(u) v^p.$$

This completes the proof the theorem.

Special case 1: Now putting k=1 in our Theorem 3we get the following result on generalized Laguerre polynomials:

Result 1: If

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n L_{n+r}^{(\alpha-n)}(x) g_n(u) w^n, \tag{2.9}$$

where $g_n(u)$ is an arbitrary polynomial of degree n, then

$$(1=t)^{\alpha} \exp(-xt) G\left[x(1+t),\, u, \frac{vt}{1+t}\right]$$

$$= \sum_{n=0}^{\infty} \sigma_n(u,v) L_{n+r}^{(\alpha-n)}(\mathrm{x}) t^n, \qquad (2.10)$$

where

$$\sigma_n(u,v) = \sum_{p=0}^{n} a_p \binom{n+r}{p+r} g_p(u) v^p .$$

which is also found derived in[4,5].

To prove the Theorem 4, we shall take help of the following generating function[7]:

$$(1+kt)^{\frac{\alpha+1-k}{k}} \exp\left[x - x(1+kt)^{\frac{1}{k}}\right] Y_{n+r}^{\alpha}\left[x(1+kt)^{\frac{1}{k}}, k\right]$$

$$= \sum_{m=0}^{\infty} k^m \binom{n+r+m}{m} Y_{n+r+m}^{\alpha-mk}(x;k) t^m, \quad (2.11)$$

3. Proof of theorem 4 :

$$R.H.S. = \sum_{n=0}^{\infty} \sigma_n(x,u,v) t^n$$

$$= \sum_{p=0}^{\infty} a_p\, g_p(u)(vt)^p \sum_{n=0}^{\infty} \binom{p+r+n}{n} Y_{p+r+n}^{\alpha-nk}(x;k)^n$$

$$= \sum_{p=0}^{\infty} a_p\, g_p(u)(vt)^p (1+kt)^{\frac{1+\alpha-k}{k}}$$

$$exp\left\{x - x(1+kt)^{\frac{1}{k}}\right\} Y_{p+r}^{\alpha}\left[x(1+kt)^{\frac{1}{k}}; k\right]$$

[*using* (2.11)]

$$= (1+kt)^{\frac{1+\alpha-k}{k}} exp\left\{x - x(1+kt)^{\frac{1}{k}}\right\}$$

$$\sum_{p=0}^{\infty} a_p\, Y_{p+r}^{\alpha}\left[x(1+kt)^{\frac{1}{k}}; k\right] g_p(u)(vt)^p$$

$$= (1+kt)^{\frac{1+\alpha-k}{k}} exp\left\{x - x(1+kt)^{\frac{1}{k}}\right\} G\left[x(1+kt)^{\frac{1}{k}}; u, vt\right],$$

[*using* (1.7)]

$= L.H.S.,$

which is Theorem 4.

Special case 2: Now putting k = 1 in our Theorem 4 we get the following result on generalized Laguerre polynomials:

Result 2: If

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n\, L_{n+r}^{(\alpha)}(\mathrm{x}) g_n(u) w^n, \quad (3.1)$$

where $g_n(u)$ is an arbitrary polynomial of degree n, then

$$(1+t)^{\alpha} \exp(-xt)\ G(x(1+t),\ u,\ vt) = \sum_{n=0}^{\infty} \sigma_n(x,v,u) t^n, \quad (3.2)$$

where

$$\sigma_n(x,u,v) \sum_{p=0}^{n} a_p \binom{n+r}{p+r} L_{n+r}^{(\alpha-n+p)}(\mathrm{x}) g_p(u) v^p,$$

which is found derived in[4].

Corollary 1: If we put r=0 in Theorem 4, then we get the following theorem:

Theorem 5: If

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n\, L_n^{\alpha}(\mathrm{x};\mathrm{k}) g_n(u) w^n \quad (3.3)$$

then

$$(1+kt)^{\frac{1+\alpha-k}{k}} exp\left\{x - x(1+kt)^{\frac{1}{k}}\right\} G\left[x(1+kt)^{\frac{1}{k}}; u, vt\right]$$

$$= \sum_{n=0}^{\infty} \sigma_n(x,u,v)t^n, \qquad (3.4)$$

where

$$\sigma_n(x,u,v) \sum_{p=0}^{n} a_p k^{n-p} \binom{n}{p} Y_n^{\alpha-nk+pk}(\mathrm{x};\mathrm{k})\, g_p(u) v^p.$$

Special case 3: Now putting k = 1 in our Theorem 5 we get the following result on generalized Laguerre polynomials:

Result 3: If

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n\, L_n^{(\alpha)}(\mathrm{x})\, g_n(u) w^n \qquad (3.5)$$

then

$$(1+t)^{\alpha} \exp(-xt)\, G(x(1+t), u, vt) = \sum_{n=0}^{\infty} \sigma_n(x,u,v)t^n, \qquad (3.6)$$

where

$$\sigma_n(x,u,v) \sum_{p=0}^{n} a_p \binom{n}{p} L_n^{(\alpha-n+p)}(\mathrm{x})\, g_p(u) v^p,$$

which is found derived in[4,6].

4. Applications: Some interesting applications of the Theorem 5 are given below:

As an application of Theorem 5, we consider the following generating relation[3]:

$$\sum_{n=0}^{\infty} \frac{n!}{\Gamma(\beta+nl+1)} Y_n^{\alpha}(x;k) Z_n^{\beta}(y;l) t^n$$

$$= (1-t)^{\frac{(1+\alpha)}{k}} \exp\left\{x - x(1-t)^{-\frac{1}{k}}\right\} H\left[x(1-t)^{-\frac{1}{k}}, \frac{-y^l t}{1-t}\right], \qquad (4.1)$$

where

$$H[x,t] \sum_{n=0}^{\infty} \frac{1}{\Gamma(\beta+nl+1)} Y_n^{\alpha}(x;k) t^n.$$

If in our theorem, we take

$$a_n = \frac{n!}{\Gamma(\beta+nl+1)}, \quad and \quad g_n(u) = Z_n^{\beta}(u;l)$$

then

$$G(x,u,w) = (1-w)^{-\frac{(1+\alpha)}{k}} exp\left\{x - x(1+w)^{-\frac{1}{k}}\right\}$$

$$H\left[x(1-w)^{-\frac{1}{k}}, \frac{-u^l w}{1-w}\right].$$

Therefore by the application of our Theorem 5 we get the following generalization of the result (4.1):

$$(1+kt)^{\frac{1+\alpha-k}{k}} (1-vt)^{-\frac{(1+\alpha)}{k}} exp\left\{x - x(1+kt)^{\frac{1}{k}} (1-vt)^{-\frac{1}{k}}\right\}$$

$$\times H\left[x(1-kt)^{\frac{1}{k}} (1-vt)^{-\frac{1}{k}}, \frac{-u^l vt}{1-vt}\right]$$

$$= \sum_{n=0}^{n} \sigma_n(x,u,v)t^n, \qquad (4.2)$$

where

$$\sigma_n(x,u,v) = \sum_{p=0}^{n} a_p k^{n-p} \binom{n}{p} Y_n^{\alpha-nk+pk}(\mathrm{x};\mathrm{k})\, g_p(u) v^p.$$

For $k = l = 1$ and $\alpha = \beta$, the generating relation (4.1) is converted to Hille-Hardy formula which can be easily generalized by result 3.

5. Conclusion

From the above discussion, it is clear that whenever one knows a bilateral generating relation of the form (1.5, 1.7) then the corresponding mixed trilateral generating relation can at once be written down from (1.6, 1.8). So one can get a large number of mixed trilateral generating relations by attributing different suitable values to in (1.5,1.7).

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