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Fixed Point Theorems and Existence of Fixed Points in Complete G –Metric Spaces

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Abstract

The main purpose of this paper is to prove some fixed point results for mappings satisfying various contractive conditions on Complete *G*-Metric spaces. We also prove the uniqueness of such fixed points as well as we showed these mappings are G-continuous on such fixed points.

Key words: G-metric spaces, symmetric G-metric spaces,

1. Introduction

The study of metric ûxed point theory has been researched extensively in the past decades, since ûxed point theory plays a major role in mathematics and applied sciences, such as optimization, mathematical models, and economic theories.

Different mathematicians tried to generalize the usual notion of metric space (X,d) such as Gahler[3,4] and Dhage[1,2] to extend known metric space theorems in more general setting, but different authors proved that these attempts are invalid.

In 2005, Mustafa and Sims[5] introduced a new structure of generalized metric spaces which are called G-metric spaces as generalization of metric space (X,d) to develop and introduce a new ûxed point theory for various mappings in this new structure.

2. Preliminaries :

Definition 2.1[5]*:* Let X be a non empty set, and let G: X × X × X → R^+ be a function satisfying the following axioms:

(G_1) $G(x,y,z) = 0$ *if* $x = y = z$,

(G_2) $0 < G(x,x,y)$, *for all* $x, y \in X$, *with* $x \neq y$,

(G_3) $G(x,x,y) \leq G(x,y,z)$, *for all* $x, y, z \in X$, *with* $z \neq y$,

(G_4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \cdots$ (symmetry in all three variables),

(G_5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$, *for all* $x, y, z, a \in X$, (Rectangle inequality).

then the function is called a generalized metric or more specifically a G-metric on X, and the pair (*X*,*G*) is called a G-metric space.

Example 2.2: Let R be the set of all real numbers. Define $G: R \times R \times R \to R^+$ by $G(x,y,z) = |x-y| + |y-z| + |z-x|$, *for all* $x, y, z \in X$. Then it is clear that (*R*,*G*) is a G-metric space.

Proposition 2.3[5]*:* Let (*R*,*G*) be a *G*-metric space. Then for any *x*,*y*,*z* and $a \in X$, it follows that

(1) *If* $G(x,y,z) = 0$, *then* $x = y = z$,

(2) $G(x,y,z) \leq G(x,x,y) + G(x,x,z)$,

(3) $G(x,y,y) \leq 2G(y,x,x)$,

(4) $G(x,y,z) \leq G(x,a,z) + G(a,y,z)$,

(5) $G(x,y,z) \leq (\frac{2}{3})(G(x,y,a) + G(x,a,z) + G(a,y,z))$,

(6) $G(x,y,z) \leq (G(x,a,a) + G(y,a,a) + G(z,a,a))$.

Definition 2.4. Let (*X*,*G*) and (*X'*,*G'*) be *G*-metric spaces and let $f : (X,G) \to (X',G')$ be a function, then f is said to be G-continuous at a point a ∈ X if given ∈ > 0, there exists δ > 0 such that $x, y \in X$; $G(a,x,y) < \delta$ implies $G'(f(a), f(x), f(y)) < \epsilon$.

A function *f* is *G*-continuous on X if and only if it is G-continuous at all $a \in X$.

Definition 2.5[5]*:* Let (*X*,*G*) be a *G*-metric space. Then for $x_0 \in X, r > 0$, *the* $G-$*ball* with centre x_0 and radius r is $B_G(x_0, r) = \{y \in X : G(x_0, y, y) < r\}$

Proposition 2.6[5]*:* Let (*X*,*G*) be a *G*-metric space. Then for any $x_0 \in X, r > 0$ one has

(1) if $G(x_0, x, y) < r$, then $x, y \in B_G(x_0, r)$,

(2) if $y \in B_G(x_0, r)$, then there exists a δ > 0 such that $B_G(y, \delta) \subseteq B(x_0, r)$.

Definition 2.7: A *G*-metric space (*X*,*G*) is called symmetric *G*-metric space if $G(x,y,y) = G(y,x,x)$ for all $x, y \in X$ and called Nonsymmetric if it is not Symmetric.

Example 2.8 Let (*R*,*d*) be the usual metric space. Define G_s and G_m by

$G_s(x,y,z) = d(x,y) + d(y,z) + d(x,z)$,and

$G_m(x,y,z) = max\{d(x,y), d(y,z), d(x,z)\}$

for all *x*,*y*,*z* ∈ *R*. Then (*R*,G_s) and (*R*,G_m) are symmetric G-metric spaces.

Example 2.9. Let $X = \{a,b,c\}$ and define $G : X \times X \times X \longrightarrow R^+$ by,

$G(x,y,z) = 0$ *if* $x = y = z$

$G(a,b,b) = G(b,a,a) = 22$

$G(a,c,c) = G(c,a,a) = 27$

$G(b,c,c) = G(c,b,b) = 30$,

$G(a,b,c) = 35$

extended by symmetry in the variables. It is easily verified that G is a symmetric G-metric, but $G \neq G_s$ or G_m for any underlying metric

Proposition 2.10[5]*:* Every G-metric space (X,G) will define a metric space (X,d_G) by

$$d_G(x,y) = G(x,y,y) + G(y,x,x), \quad \forall x,y \in X \tag{2.1.1}$$

If (X,G) is a symmetric G-metric space, then

$$d_G(x,y) = 2G(x,y,y), \forall x,y \in X \tag{2.1.2}$$

However, if (X,G) is not symmetric, then it holds by the G-metric properties that

$$\frac{3}{2}G(x,y,y) \leq d_G(x,y) \leq 3G(x,y,y), \quad \forall x,y \in X \tag{2.1.3}$$

and that in general these inequalities cannot be improved.

3. The Main Results

In this section we will prove several theorems in each of which we have omitted the completeness property of G-metric space and we have obtained the same conclusion as in complete G-metric space, but with assumed sufficient conditions.

Theorem 3.1: Let (X,G) be a G-metric space and let $T: X \to X$ be a mapping such that T satisfies that

$$G(Tx,Ty,Tz) \leq aG(x,Tx,Tx) + bG(y,Ty,Ty) + cG(z,Tz,Tz) \tag{3.1.1}$$

for all $x,y,z \in X$ where $0 < a + b + c < 1$,

Then T has a unique fixed point $u \in X$ and T is G-continuous at u.

Proof: Suppose that T satisfies condition (1), then for all $x,y \in X$, we have

$$G(Tx,Ty,Ty) \leq aG(x,Tx,Tx) + (b+c)G(y,Ty,Ty)$$
$$G(Ty,Tx,Tx) \leq aG(y,Tx,Tx) + (b+c)G(x,Tx,Tx) \tag{3.1.2}$$

Suppose that (X,G) is symmetric, then by definition of metric (X,d_G) and (1.2), we get

$$d_G(Tx,Ty) \leq \frac{(a+b+c)}{2} d_G(x,Tx) + \frac{(a+b+c)}{2} d_G(y,Ty) \forall x,y \in X \tag{3.1.3}$$

In this line, since $0 < a + b + c < 1$, then the existence and uniqueness of the fixed point follows from well-known theorem in metric space (X,d_G) (see[10]).

However (X,d_G) is not symmetric then by definition of metric (X,d_G) and (1.3), we get

$$d_G(Tx,Ty) \leq \frac{2}{3}(a+b+c)d_G(x,Tx) + \frac{2}{3}(a+b+c)d_G(y,Ty) \tag{3.1.4}$$

for all $x,y \in X$, then the metric condition gives no information about this map since $< \frac{2}{3}(a+b+c) + \frac{2}{3}(a+b+c)$ need not be less than 1. But this can be proved by G-metric.

Let $x_0 \in X$ be an arbitrary point, and define the sequence $\{x_n\}$ by $x_n \in T^n(x_0)$. By (1), we have

$$G(x_n, x_{n+1}, x_{n+1}) = G(T^n(x_o), T^{n+1}(x_o), T^{n+1}(x_o)$$
$$\leq G\big(T^{n-1}(x_o), T^n(x_o), T^n(x_o)\big) + bG(T^n(x_o),$$
$$T^{n+1}(x_o), T^{n+1}(x_o) + cG(T^n(x_o), T^{n+1}(x_o), T^{n+1}(x_o))$$
$$G(x_n, x_{n+1}, x_{n+1}) \leq aG(x_{n-1}, x_n, x_n) + (b+c)G(x_n, x_{n+1}, x_{n+1}) \tag{3.1.5}$$

Then $G(x_n, x_{n+1}, x_{n+1}) \leq \dfrac{a}{1-(b+c)} G(x_{n-1}, x_n, x_n)$ (3.1.6)

Let $q = \dfrac{a}{1-(b+c)}$, then $0 \leq q < 1$ *since* $0 \leq a + b + c < 1$.

So, $G(x_n, x_{n+1}, x_{n+1}) \leq qG(x_{n-1}, x_n, x_n)$(3.1.7) Continuing in the same argument, we will get

$$G(x_n, x_{n+1}, x_{n+1}) \leq q^n G(x_0, x_1, x_1) \qquad (3.1.8)$$

Moreover, for all $n,m \in N; n < m$, we have by rectangle inequality that

$$G(x_n, x_m, x_m) \leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2})$$
$$+ G(x_{n+2}, x_{n+3}, x_{n+3}) \dots\dots + G(x_{m-1}, x_m, x_m)$$
$$\leq (q^n + q^{n+1} + \cdots \dots\dots\dots\dots\dots\dots + q^{m-1}) G(x_0, x_1, x_1)$$

$$\leq \frac{q^n}{1-q} G(x_0, x_1, x_1) \qquad (3.1.9)$$

and solim $G(x_n,x_m,x_m)=0$ as $n,m \to \infty$. Thus $\{x_n\}$ is G-Cauchy sequence. Due to the completeness of (X, G) there exists $u \in X$ such that $\{x_n\}$ is G-converge to u.

Suppose $Tu \neq u$, then

$$G(x_n, T(u), T(u)) \leq aG(x_{n-1,} x_n, x_n) + (b+c)G(u, T(u), T(u)) \qquad (3.1.10)$$

taking the limit as $n \to \infty$, and using the fact that the function is G continuous, then

$$G(u, T(u), T(u)) \leq (b+c)G(u, T(u), T(u)).$$

This contradiction implies that $u=Tu$.

To prove uniqueness, suppose that $u \neq v$ such that $Tv \neq v$, then

$$G(u, v, v) \leq aG(u, Tu, Tu) + (b+c)G(v, Tv, Tv) = 0$$

which implies that $u=v$.

To show that T is G-continuous at u, let $(y_n) \subseteq X$ be a sequence such that lim $(y_n)=u$. We can deduce that

$$G(u, T(y_n), T(y_n)) \leq aG(u, Tu, Tu) + (b+c)$$
$$G(y_n, T(y_n), T(y_n)) = (b+c)G(y_n, T(y_n), T(y_n))$$

andsince

$$G(y_n, T(y_n), T(y_n)) \leq G(y_n, u, u) + G(u, T(y_n), T(y_n)),$$

we have

$$G(u, T(y_n), T(y_n)) \leq (b+c)(G(y_n, u, u) + G(u, T(y_n), T(y_n)),$$

$$G(u, T(y_n), T(y_n)) \leq \frac{b+c}{1-(b+c)} G(y_n, u, u)$$

taking the limit as $n \to \infty$, from which we see that $G(u,T(y_n),T(y_n)) \to 0$ and so, by proposition 2.3, $T(y_n)) \to u=Tu$. It is proved that T is G-continuous at u.

Theo $T^{ni}(x) \to u$ and $T^{ni+1}(x)) \to Tu$, *rem 3.2:* Let (X,G) be a complete G-metric space and let $T: X \to X$ be a mapping satisfies the following condition for all $x,y,z \in X$

(B_1) $G(Tx, Ty, Tz) \leq aG(x, Tx, Tx) + bG(y, Ty, Ty) + cG(z, Tz, Tz) + dG(x, y, z)$, where $0 \leq a+$, $b+c+d<1$,

(B_2) T is G-continuous at a point $u \in X$

(B_2) there is $x \in X$; $\{T^n(x)\}$ has a subsequence $\{T^{ni}(x)\}$ G-converges to u. Then u is a unique fixed point (i. e. Tu = u).

Proof: G-continuity of T at u implies that, $\{T^{ni+1}(x)\}$ G-convergent to $T(u)$. Suppose $T(u) \neq u$, consider the two G-open balls $B_1=B(u,\in)$ and $B_2=B(u,\varepsilon)$ where $\epsilon < \frac{1}{6}$=min $\{G(u,Tu, Tu), G(Tu, u, u)\}$.

Since $T^{ni}(x) \to u$ and $T^{ni+1}(x)) \to Tu$, then there exist $N_1 \in N$ such that if i $> N_1$ implies $T^{ni}(x) \in B_1$ and $T^{ni+1}(x) \in B_2$. Hence our assumption implies that we must have

$$G\left(T^{ni}(x), T^{ni+1}(x), T^{ni+1}(x)\right) > \epsilon, \quad \forall i > N_1 \qquad (3.2.1)$$

On the other hand we have from (B_1),

$G\left(T^{ni+1}(x), T^{ni+2}(x), T^{ni+3}(x)\right) \leq aG\left(T^{ni}(x), T^{ni+1}(x),\right.$ $\left.T^{ni+2}(x)\right) + bG\left(T^{ni}(x), T^{ni}(x), T^{ni+1}(x)\right) +$ $cG\left(T^{ni+1}(x), T^{ni+1}(x), T^{ni+2}(x)\right) + dG\left(T^{ni+2}(x),\right.$ $\left.T^{ni+2}(x), T^{ni+3}(x)\right)$ (3.2.2)

But by axioms of G-metric (G_3), we have

$G\left(T^{ni}(x), T^{ni}(x), T^{ni+1}(x)\right) \leq G\left(T^{ni+1}(x), T^{ni+2}(x), T^{ni+3}(x)\right)$ (3.2.3)

$G\left(T^{ni+1}(x), T^{ni+1}(x), T^{ni+2}(x)\right) \leq G\left(T^{ni+1}(x),\right.$ $\left.T^{ni+2}(x), T^{ni+3}(x)\right)$ (3.2.4)

$G\left(T^{ni+2}(x), T^{ni+2}(x), T^{ni+3}(x)\right) \leq G\left(T^{ni+1}(x),\right.$ $\left.T^{ni+2}(x), T^{ni+3}(x)\right)$ (3.2.5)

From (3), (4) and (5) , we see (2) becomes

$G\left(T^{ni+1}(x), T^{ni+2}(x), T^{ni+3}(x)\right) \leq q\, G\left(T^{ni}(x),\right.$ $\left.T^{ni+1}(x), T^{ni+2}(x)\right)$ (3.2.6)

where $q = {}^{a}/_{(1-(b+c+d))}$ and $q < 1$, since $0 < a + b + c + d < 1$.

For $> j > N_1$, and by repeated applications of (6) we have

$G(T^{n_l}(x), T^{n_l+1}(x), T^{n_l+2}(x)) \leq qG(T^{n_l-1}(x), T^{n_l}(x), T^{n_l+1}(x))$
$\leq q^2 G(T^{n_l-2}(x), T^{n_l-1}(x), T^{n_l}(x))$
$\leq \cdots \ldots \leq q^{n_l-n_j} G(T^{n_j}(x), T^{n_j+1}(x), T^{n_j+2}(x))$ (3.2.7)

So, as $l \to \infty$ we have $\lim G\left(T^{n_l}(x), T^{n_l+1}(x),\right.$ $\left.T^{n_l+2}(x)\right) \leq 0$ which contradict (1), hence $Tu=u$. Suppose there is $v \in X; Tv = v$, then from (B_1), we have

$G(u,v,v) = G(Tu,Tv,Tv) \leq aG(u,v,v)$
$+ bG(u,u,Tu) + (c+d)G(v,v,Tv)$
$G(u,v,v) \leq 0$ (3.2.8)

This proves the uniqueness of u.

Theorem 3.3: Let (X,G) be a complete G-metric space and let $T{:}X \to X$ be a mapping satisfies the following condition for all $x,y,z \in X$

$G(Tx,Ty,Tz) \leq aG(x,Tx,Tx) + bG(y,Ty,Ty) + cG(z,Tz,Tz)$ *where* $0 \leq a+b+c < 1$, *then* T *has a unique fixed point, say* u *and* T *is* $G-$*continuous at* u.

Example 3.4: Let $X = [0,1), Tx = x/4$ *and* $G(x,y,z) = max\{|x-y|, |y-z|, |x-z|\}$.

Then (X,G) is G-metric space but not complete, since the sequence $x_n = 1 - \frac{1}{n}$ is G-Cauchy which is not G-convergent in (X,G).

Theorem 3.5: Let (X,G) be a G-metric space and let $T : X \to X$ be a G- continuous mapping satisfies the following conditions:

(B_1) $G(Tx,Ty,Tz) \leq k\{G(x,Tx,Tx) + G(y,Ty,Ty) + G(z,Tz,Tz)\}$ *for all* $x,y,z \in M$, where M is an every where dense subset of X (with respect the topology of G-metric convergence) and $0<k<1/6$,

(B_2) there is $x \in X$ such that $\{T^n(x)\} \to x_0$. Then x_0 is unique fixed point.

Corollary 3.6: Let (X,G) be G-metric space and let $T : X \to X$ be a mapping such that T satisfies that

(C_1) $G(Tx,Ty,Ty) \leq aG(x,Tx,Tx) + bG(y,Ty,Ty)$ *for all* $x,y \in X$ *where* $0 < a+b < 1$,

(C_2) T *is* $G-$continuous at a point $u \in X$,

(C_3) there is $x \in X$; $\{T^{ni}(x)\}$ has a subsequence $\{T^{ni}(x)\}$ G-converges to u. Then u is a unique fixed point ($i.\ e.$ Tu = u).

Corollary 3.7: Let (X,G) be G-metric space and let $T : X \to X$ be a G-continuous mapping satisfies that

(D_1) $G(Tx,Ty,Ty) \leq k\{G(x,Tx,Tx) + G(y,Ty,Ty)\}$ for all $x, y \in M$ where M is an every where dense subset of X (with respect the topology of G-metric convergence) and $0<k<1/6$,

(D_2) there is $x \in X$ such that $\{T^n(x)\} \to x_0$. Then x_0 is unique fixed point.

Corollary 3.8: Let (X,G) be G-metric space and let $T : X \to X$ be a mapping such that T satisfies that

(E_1) $G(Tx,Ty,Ty) \leq kG(x,y,y)$ *for all* $x, y \in X$ *where* $0 < k < 1/4$,

(E_2) T is G-continuous at a point $u \in X$,

(E_3) there is $u \in X$; $\{T^n(x)\}$ has a subsequence $\{T^{n_i}(x)\}$ G-converges to u. Then u is a unique fixed point.

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