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A Common Fixed point Theorem for fuzzy Type mapping in Hilbert space

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Abstract

In this paper we worked out fixed point theorem for fuzzy mapping on Hilbert space.

Key words: fuzzy mapping, Hilbert space, fixed point, approximate quantity.

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1. Introduction

In this paper, we prove fixed point theorem of fuzzy mapping as introduced by Heilpern applied to Hilbert spaces[1-3].

2. *Preliminaries :*

Each fuzzy set in a universe $A: M \to [0, 1]$ which is called the member ship function of that fuzzy set as usual the set "supp A" $=\{x \in M : A(x) \neq 0\}$

Let H be a Hilbert space and F(H) be a collection of all fuzzy set in H. let $A \in F(H)$ and $\propto \in [0,1]$ the $\propto$ level set of A, denoted by $A_\propto$ is defined as

$$A_\propto = \{x: A(x) \geq \propto\} \quad \text{if } \alpha \in (0,1]$$
$$A_0 = \{x: A(x) > 0\}$$

Definition 1: A mapping F from the set H onto W (H) is said to be a fuzzy mapping. Any $x \in H$ is called a fixed point of a mapping $F: H \to W(H)$ if

$$\{x\} \subset \{Fx\}$$

Where $\{x\}$ is the fuzzy set with a membership function equal to the characteristic function of the crist set $\{x\}$.

Definition 2: let $A \in W(H)$ and $\propto \in [0,1]$, then

(i) $P_{\propto}(A,B) = \inf_{x\in A, Y\in B,}\|x-y\|$

(ii) $D_{\propto}(A,B) = dist(A_{\propto,}B_{\propto,})$,where "dist" denotes the housdorff metric between $A_{\propto}$ and $B_{\propto,}$

(iii) $D(A,B) \quad = \sup_{\alpha} D_{\propto}(A,B)$

(iv) $P(A,B) \quad = \sup_{\alpha} P_{\propto}(A,B)$

It is to be noted that for any is a nondecreasing as well as continuous function[4-6]

Definition 2. A fuzzy subset A of H is said to be an approximate quantity iff its $\propto$ level set is a nonfuzzy compact convex subset of H for each $\propto$[0,1] and sup$x\in H$ $A(x) = 1$

Definition 3. Let $A, B \in W(H)$.An approximate quantity A is said to be more accurate than B, denoted by $A \subset B$, iff $A(x) < B(x)$ for each $x \in H$. The relation induces a partial ordering on W(H)

We shall use the following lemma

Lemma 1. Let $x \in H$, $A \in W(H)$ then $\{x\} \subset A$ if and only if $P_{\alpha}(x,A) = 0$ for each $\propto \in [0,1]$

Lemma 2. $P_{\propto}(x,A) \leq \|x-y\| + P_{\alpha}(y,A)$ for any $x, y \in H$

Lemma 3. if $\{x_0\} \subset A$, then $P_{\alpha}(x_0, B) \leq D_{\alpha}(A,B)$ for each $B\in W(H)$.

$$\|x-y\|^2 + b\{{P_{\alpha}}^2(x,Sx) + {P_{\alpha}}^2(y,Ty)\} + c\,\{{P_{\alpha}}^2(x,Ty) + {P_{\alpha}}^2(y,Sx)\}$$

3. **Main results**

Theorem 1. Let H be a Hilbert space S and T are fuzzy mappings from H into W (H) satisfying

$$D^2(Sx,Ty) \leq a\|x-y\|^2 + b\{{P_{\alpha}}^2(x,Sx) + {P_{\alpha}}^2(y,Ty)\} + c\,\{{P_{\alpha}}^2(x,Ty) + {P_{\alpha}}^2(y,Sx)\} \tag{1}$$

For all x,y, in H and for all $\alpha\in[0,1]$ and a, b, c are nonnegative numbers satisfying

$$a + b + c < 1 \tag{2}$$

Then there exists a point z in H such that

$$\{z\} \subset Sz, \cap Tz$$

Proof. Let $x_0 \in H$. we construct the sequence $\{\{x_n\}\}$ as follows

$$\{x_1\} \subset Sx_{0,}\{x_2\} \subset Tx_1, \dots, \{x_{2n+1}\} \subset Sx_{2n,}\,\{x_{2n+2}\} \subset Tx_{2n+1}$$

And

$$\|x_i - x_{i+1}\| \leq D(Sx_{i-1}, Tx_i)\; i = 1,2 \dots$$

Now,

$$\begin{aligned}
&\left\|x_{2n,} - x_{2n+1}\right\|^2 \leq D^2\,(Sx_{2n,}\,Tx_{2n-1})\\
&\leq a\left\|x_{2n,} - x_{2n-1}\right\|^2 + b\{{P_{\alpha}}^2(x_{2n\,,}\,Sx_{2n}) + {P_{\alpha}}^2(x_{2n-1\,,}\,Tx_{2n-1})\}\\
&\qquad + c\{{P_{\alpha}}^2(x_{2n\,,}\,Tx_{2n-1}) + {P_{\alpha}}^2(x_{2n-1\,,}\,Sx_{2n})\}\\
&\leq a\left\|x_{2n,} - x_{2n-1}\right\|^2 + b\left[\left\|x_{2n,} - x_{2n+1}\right\|^2 + \left\|x_{2n,-1} - x_{2n}\right\|^2\right]\\
&\qquad + c\left[\left\|x_{2n,} - x_{2n}\right\|\left\|x_{2n-1,} - x_{2n+1}\right\|^2\right]
\end{aligned}$$

$$\leq a\|x_{2n,}-x_{2n-1}\|^2 + b\left[\|x_{2n,}-x_{2n+1}\|^2 + \|x_{2n,-1}-x_{2n}\|^2\right]$$
$$+\,\mathrm{c}\left[\|x_{2n,}-x_{2n}\|\|x_{2n-1,}-x_{2n+1}\|^2\right]$$
$$\leq a\|x_{2n,}-x_{2n-1}\|^2 + \mathrm{b}\left[\|x_{2n,}-x_{2n+1}\|^2 + \|x_{2n,-1}-x_{2n}\|^2\right]$$
$$+\,\mathrm{c}\left[\|x_{2n,}-x_{2n+1}\|^2 + \|x_{2n,-1}-x_{2n}\|^2\right]$$
$$((1-b-\mathrm{c})\|x_{2n,}-x_{2n+1}\|^2 \leq (1+\mathrm{b}+\mathrm{c})\|x_{2n,}-x_{2n-1}\|^2$$

Which gives

$$\|x_{2n,}-x_{2n+1}\|^2 \leq p_1\,\|x_{2n,}-x_{2n-1}\|^2$$

Where $0 < p_1 = \frac{(\mathrm{a+b+c})}{(1-\mathrm{b}-\mathrm{c})} < 1$

Again

$$\|x_{2n-1,}-x_{2n}\|^2 \leq \mathrm{D}^2(\mathrm{S}x_{2n-2,}\,\mathrm{T}x_{2n-1})$$

$$\leq a\|x_{2n-2,}-x_{2n-1}\|^2 + b\{P_\alpha{}^2(x_{2n-2,}\,Sx_{2n-2}) + P_\alpha{}^2(x_{2n-1}\,,Tx_{2n-1})\} + \mathrm{c}\{P_\alpha{}^2(x_{2n-2}\,,Tx_{2n-1}) + P_\alpha{}^2(x_{2n-1}\,,Sx_{2n-2})\}$$

$$\leq a\|x_{2n-2,}-x_{2n-1}\|^2 + \mathrm{b}\left[\|x_{2n-2,}-x_{2n-1}\|^2 + \|x_{2n,-1}-x_{2n}\|^2\right] + \mathrm{c}\left[\|x_{2n-2,}-x_{2n}\|^2 + \|x_{2n,-1}-x_{2n-1}\|^2\right]$$

$$\leq a\|x_{2n-2,}-x_{2n-1}\|^2 + \mathrm{b}\left[\|x_{2n-2,}-x_{2n-1}\|^2 + \|x_{2n,-1}-x_{2n}\|^2\right]$$
$$+\,c\left[\|x_{2n-2,}-x_{2n-1}\|^2 + \|x_{2n,-1}-x_{2n}\|^2\right]$$

$$\|x_{2n-1,}-x_{2n}\|^2 \leq \frac{(\mathrm{a+b+c})}{(1-\mathrm{b}-\mathrm{c})}\|x_{2n-2,}-x_{2n-1}\|^2$$

Which gives

$$\|x_{2n-1,}-x_{2n}\|^2 \leq p_2\,\|x_{2n-2,}-x_{2n-1}\|^2$$

Where $0 < p_2 = \frac{(\mathrm{a+b+c})}{(1-\mathrm{b}-\mathrm{c})} < 1$

Choosing p = max $\{p_{1,}p_{1,}\}$,, it follows that

$$\left\|x_{n+1,} - x_n\right\|^2 \leq P\left\|x_{n,} - x_{n-1}\right\|^2$$

Where 0<p<1.

Hence $\{x_n\}$ is a Cauchy sequence in H and therefore it converges to a limit in H. we assume $\lim_{n\to\infty} x_n = z$

Again using lemma 3 and for all $\alpha \in [0,1]$

$P_\alpha{}^2(x_{2n+2,} Sz) \leq D^2$ (T$x_{2n+1,}$ Sz)

$\leq a\|x_{2n+1} - z\|^2 + b\{P_\alpha{}^2(x_{2n+1,} Tx_{2n+1}) + P_\alpha{}^2(z, Sz)\} + c\{P_\alpha{}^2(z, Sz) + P_\alpha{}^2(z, Tx_{2n+1})\}$

$\leq a\left\|x_{2n+1,} - z\right\|^2 + b\left\{\left\|x_{2n+1,} - x_{2n+2}\right\|^2 + P_\alpha{}^2(z, Sz)\right\} + c\{P_\alpha{}^2(z, Sz) + \|z - x_{2n+2}\|^2\}+$

Making $n\to\infty$ and using the fact that p_α is continuous

$$P_\alpha{}^2(z, Sz) \leq b\{P_\alpha{}^2(z, Sz)\} + c\{P_\alpha{}^2(z, Sz)\}$$

$$P_\alpha{}^2(z\,Sz) \leq (b+c)P_\alpha{}^2(z\,Sz)$$

As $(b+c)<1$ it follows that $P_\alpha{}^2(z\,Sz) = 0$ hence by lemma1, $\{z\} \subset Sz$. Similarly, $\{z\} \subset Tz$.. Hence $\{z\} \subset Sz \cap Tz$.

References

1. P. N. Dutta and B. S. Choudhury Fuzzy mapping and Fixed point theorems, *7*, 91-96 (2002).
2. R.K. Bose, D. Sahani, .Fuzzy mappings and fixed point theorem, *Fuzzy Sets and Systems 21*, 53-58 (1987).
3. S. Heilpern, Fuzzy mapping and Fixed point theorems, *J. Math. Anal. Appl. 83*, 566-569 (1981).
4. B.E. Rhoades, Fixed point of some Fuzzy mapping, *Soochow Journal of Mathematics. 22*, 111-115 (1996).
5. T. Som, R.N. Mukherjee, some fixed point of Fuzzy mapping, *Fuzzy Sets and System 33*, 213-219 (1989).
6. S.L. Singh, R. Talwar, Fixed point of some Fuzzy mappings, *Soochow Journal of Mathematics. 19*, 95-102 (1993).