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Quasi-P Normal operators linear operators on Hilbert space for which T+T* And T*T commute

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Abstract

Binormal operator: We say that an operator T on a Hilbert Space H is bi-normal if TT* and T*T commute

i.e $[T^*T, TT^*] = 0$

I.e $T^*T\ TT^* = TT^*T^*T = 0$

Where T* is the ad joint of T

Quasi normal operator : We say that an operator T on a Hilbert Space H is quasi normal operator if T and T*T commute

i.e $[T, T^*T] = 0$

I.e $TT^*T = T^*TT$

Quasi– P-normal operator : An operator T on a Hilbert Space H is said to be quasi P-normal operator if T+T* and T*T Commute

i.e $[T+T^*, T^*T] = 0$

I.e $(T+T^*)T^*T = T^*T(T+T^*)$

I.e $TT^*T+T^*T^*T = T^*TT+T^*TT^*$

Introduction

Theorem 1: If Tis a Quasi P normal operator and λ is any scalar which is real then λ T is also a quasi P normal operator[1-3].

Proof : Since T is a quasi P normal operator

$[T+T^*, T^*T\} = 0$

i.e $T^*T(T+T^*) = (T+T^*)T^*T = 0$ (1)

If λ be any scalar which is real, then $(\lambda T)^* = \lambda T^* = \lambda T^*$

Now $(\lambda T)^*(\lambda T)(\lambda T+(\lambda T)^*) = \lambda^3 T^{*}+ T(T+T^*)$ (2)

And $(\lambda T+(\lambda T)^*)(\lambda T)^*(\lambda T) = \lambda^3 (T+T^*)T^*T$ (3)

From (1) (2) and (3) we see that λ T is quasi P normal

Theorem – 2: If T is a quasi P normal operator which is a normal operator also, Then T^* is also a quasi P- normal operator.

Proof: Since T is a quasi –P normal operator.

$(T+T^*)\,T^*T = T^*T(T+T^*)$ (1)

Since T is normal therefore

$T^*T = TT^*$ (2)

Now substituting T^* for T in (1) we have $(T^*+(T^*)^*)\ (T^*)^*\,T^*$

$= (T^*+T)TT^*$ (3)

And $(T^*)^*T(T^*+(T^*)^*)\text{-}=TT^*\,(T^*+T)$

$= T^*T(T+T^*)$ (4)

By (2)(3)(4)$(T^*+T^{**})\,T^{**}T^*= T^{**}T^*(T^*+T^{**})$

Hence T^* is also quasi P normal operator

Theorem 3: If T is a self ad-joint operator, Then T is a quasi–P normal operator.

Proof : Since T is a self ad-joint operator.

Therefore $T^* = T$

Now $T^*T\,(T+T^*) = TT\,(T+T) = 2T^3$ (1)

$(T+T^*)\,T^*T = (T+T)\,TT = 2T^3$ (2)

Hence $T^*T\,(T+T^*) = (T+T^*)\,T^*T$ (3)

So T is a quasi P normal operator.

Theorem 4: Let T be any operator on a Hilbert Space H Then

i) $T+T^*$ is quasi P normal.

ii) TT^* is quasi P-normal.

iii) T^*T is quasi P-normal.

iv) $I+T^*T$, $I+TT^*$ are also quasi P-normal.

Proof : Let $N = T+T^*$

Now $N^* = (T+T^*)^*$

$= T^*+T^{**}=T^*+T=T+T^*= N$

$\therefore$ N is a self adjoint operator. And every self ad joint operator is quasi P normal operator. Therefore $N = T+T^*$ is a quasi P normal operator.

Similarly $(TT^*)^* = T^{**}T^* = TT^*$

$(T^*T)^* = T^*T^{**} = T^*T$

$(I+ TT^*)^* = I^*+ (TT)^* = I +TT^*$

$(I+T^*T)^*=I^*+(T^*T)^*=I+T^*T$

So all the operators mentioned above are Self adjoint and therefore all the above mentioned operators are quasi P normal operator

C *or*→*Suice O and* I *are self ad joint operators Therefore O and* I *are quasi P normal operators*

Theorem 5: Let T be a quasi P normal operator on a Hilbert Space H Let S be a Self adjoint operator for which T and S Commut, then

ST is also a self adjoint operator.

Proof : Since S is a Self adjoint operator.

Therefore $S^* =S$.

Now S and T Commute.

Therefore $ST = TS$

$(ST)^* = (TS)^*$

$\Rightarrow T^*S^* = S^*T^*$

$\Rightarrow T^*S = ST^*$

Now Since T is a quasi P normal operator.

Therefore $[T+T^* ,\ T^*T] = 0$

i.e $T^*T\,(T+T^*) = (T+T^*)\,T^*T$

Now putting ST for T we have

$(ST)^*(ST)(ST+(ST)^*)= T^*S\,ST(ST+T^*S)$

$=ST^*ST(ST+ST^*) =SST^*T(T+T^*)S$

$=SS\,(T+T^*)T^*TS =(ST+ST^*)\,ST^*TS$

$=(ST +(ST)^*)\,(ST)^*ST$

$\Rightarrow$ ST is a quasi P normal operator

Theorem 6: Let T be a quasi P normal operator which, is a unitary operator also, then T^{-1} is also a quasi P normal operator

Proof : Since T is a quasi P normal operator

So, $T^*T\ (T+T^*)=(T+T^*)T^*T$

Since T is a unitary operator.

Therefore $TT^* = T^*T = I\ ,\ T^*= T^{-1}$

So $T^{**} = (T^{-1})^*\quad T = (T^{-1})^*$

Also $(T^{-1})^* = T$

$\because (T^{-1})^* = (T^*)^{-1}$

Now $T^*T\ (T+T^*) = (T+T^*)\ T^*T$

$TT^*\ (T+T^*) = (T+T^*)\ TT^*$

Or $(T^{-1})^* (T^{-1})\ ((T^{-1})^*+ T^{-1})=((T^{-1})^*+ T^{-1})$

$(T^{-1})^*\ T^{-1}$

$\Rightarrow T^{-1}$ is a quasi P normal operator[4-6].

Theorem 7: Let T=R+iS be an operator on a Hilbert Space H. for which RS=SR, then T is a quasi P normal operator if RSS= SSR

Proof : Since T is an operator for which $T=R+iS$

So $T^* = R-iS$

$T + T^* = R + iS + R-iS = 2R$

$TT^* = (R+iS)\ (R-iS) = (RR+SS) +i\ (SR - RS) = RR+SS + i\ .\ 0$

$\therefore TT^* = RR + SS$

Now $(T+T^*)\ T^*T = 2R\ (SS+RR) = 2\ (RSS+R^3)$

$T^*T(T+T^*)=(SS+RR)2R,=2(SSR+R^3)$

Since $RSS = SSR$

$\therefore T^*T\ (T+T^*) = (T+T^*)T^*T$

$\Rightarrow T$ is quasi P normal

Theorem 8 : Let M be the class of all Quasi P normal operators on a Hilbert space H, then M is a Closed Subset of B (H) contaiug the set of all self adjoint operators on the Hilbert Space H.

Proof : Let M be the set of all quasi P normal operators. We have to show that M is a Closed subset of B (H). Let <Tn> be the sequence of quasi P normal operators. Which converges on T We shall show that T M *i.e* T is a quasi P normal operator[1-6].

Now, $II\ T^*T\ (T+T^*)-(T+T^*)\ T^*T\ II$

$= II\ T^*T\ T+T^*\ TT^*- TT^*T^*T - T^*T^*T\ II$

$= II\ T^*TT - T_n^*T_n\ (T_n + T_n^*) + (T_n + T_n^*)$

$T_n^* T_n - T^*T^*T+T^*TT^*-TT^*T\ II$

$[\because T_n^* T_n\ (T_n + T_n^*) - (T_n + T_n^*)\ T_n^* T_n= 0]$

$\leq II\ T^*TT - T_n^*T_n T_n II+II\ T^*TT^* - T_n^*T_n$

$T_n^*II+ II\ T_n T_n^*T_n-TTT^*II$

$+\ II\ T_n^* T_n^*T-T^*T^*TII$

$\rightarrow 0$ as $T_n\rightarrow T$ and $n \rightarrow \infty$

$\therefore IIT^*T(T+T^*) - (T+T^*)T^*TII=0$

$\Rightarrow T^*T(T+T^*) =(T+T^*)T^*T$

$\Rightarrow$ T is a quasi P normal operator

$T\in M\ \therefore$ M is a closed subset of B(H). Since every self adjoins operator is a quasi P normal operator. So M contains all the self adjoins operators[1-4].

Theorem 9: An idempotent operator T on a Hilbert space H is binormal iff it is a quasi P normal operator.

Given that T is an idempotent operator on H.

$TT=T\ ,\ T^*T^*=T^*$

Now Suppose T is quasi P normal

$T^*T(T+T^*) = T^*TT+T^*TT^*$

$= T^*T+T^*TT^*,\ (T+T^*)T^*T= TT^*T +T^*T^*T = TT^*T +T^*T$

Since T is quasi P normal.

$\therefore (T+T^*)T^*T - T^*T\ (T+T^*) = 0$

$\Rightarrow TT^*T+T^*T^*T - T^*TT+T^*TT^*=0$

$\Rightarrow TT^*T+T^*T-T^*T+T^*TT^* =0$

$\Rightarrow TT^*T-T^*TT^*=0$

$\Rightarrow TT^*T^*T-T^*TTT^*=0$

$\Rightarrow [TT^*,T^*T] =0$

$\Rightarrow$T is binomial operator.

Conversely Suppose that T is a binormal operator

$\therefore TT^*T^*T = T^*T\, TT^*$

$\Rightarrow TT^*T = T^*TT^*$

$\Rightarrow TT^*T + T^*T = T^*TT^* + T^*T$

$\Rightarrow TT^*T + T^*T^*T = T^*TT^* + T^*TT$

$\Rightarrow (T+T^*)T^*T = T^*T(T^*+T)$

$\Rightarrow$T is quasi P normal operator.

Theorem 10 : Let T_1 and T_2 be two quasi P normal operators for which each is the adjoint of other, then $T_1 + T_2$ and $T_1 T_2^*$ are also quasi P normal operators[1-6].

Proof : Here we are given that

$T_1{}^*T_1(T_1+T_1{}^*) = (T_1+T_1)\, T_1{}^*\, T_1$

And $T_2{}^*T_2(T_2+T_2{}^*) = (T_2+T_2)\, T_2{}^*\, T_2$

Also $T_1{}^* = T_2$ and $T_2{}^* = T_1$

It is easy to verify by calculation that

$(T_1+T_2)^*(T_1+T_2)\, ((T_1+T_2)+(T_1+T_2)^*)$

$= ((T_1+T_2)+(T_1+T_2)^*)\, (T_1+T_2)^*\, (T_1+T_2)$

And $(T_1 T_2)^*\, (T_1 T_2)\, (T_1 T_2 + (T_1+T_2)^*)$

$= (T_1 T_2 + (T_1 T_2)^*)\, (T_1 T_2)^*\, (T_1 T_2)$

So, T_1+T_2 and $T_1 T_2$ are quasi P normal operators

*Theorem 11: Let T be a self adjoint operator on a Hilbert Space H. and S be any operator on H ,then S*TS is a quasi P normal operator on H.*

Proof : - Since T is self adjoint

$\therefore T^* = T$

Now $(S^*TS)^* = S^*T^*S^{**}$

$= S^*T^*S$

$= S^*TS$

*Which shows that S*TS is self adjoint, and since every self adjoin operator is quasi P normal operator, therefore S*TS is also quasi P normal*

Theorem 12: Let T=ƱP be the polar decomposition of an operator T on a Hilbert space H.

Where Ʊ is an unitary operator

Then $[T+T^*,\ T^*T]=0$ *iff* $[ƱP+PƱ^*, p^2]=0$

The null space of Ʊ is equal to the null space of P.

Proof: HereƱis an *unitary* operator.

$\therefore Ʊ^*Ʊ = ƱƱ^* = I$

Now T=ƱP is the Polar decomposition of T where U is *unitary*.

$\therefore T^* = P^*$

Now $T^*T = PƱ^*ƱP = PP = P^2$

Given N(Ʊ) = N(P)

Now [T+T* TT*] =0

$\Rightarrow [ƱP=PƱ^*, PƱ^*ƱP] = 0$

$\Rightarrow [ƱP=PƱ^*, p^2] = 0$

References

1. Simmons, G.F., Introduction to topology and modern analysis, NEW YORK : Mc Graw-Hill (1963).
2. Halmos, P.R., A Hilbert space problem book, Princeton, N.J : Van Nostrand (1967).
3. Berberian, S. K., Lectures in Functional Analysis and operator theorey : Springer Verlag New York. (1974).
4. Taylor, A.E., Introdction to Functional Analysis. NEW YORK Wlley (1958).
5. Rudin, W., Functional Analysis, New York, Mc Graw Hill (1973).
6. Maddox, I.J., Elements of functional analysis Cambridge, Cambridge University Press (1970).