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Random algebraic polynomials with non-identically distributed normal coefficients

[1]S. BAGH and [2]M.K. PAL

[1]Department of Statistics, Sambalpur University, Orissa (INDIA)
[2]Institute of Management & Information Science,
Bhubaneswar, Orissa (INDIA)
E-mail: sbagh55@hotmail.com, E-mail: manas.sbp@gmail.com

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Abstract

The asymptotic estimate for the expected number of real zeros of a random algebraic polynomial $Q_n(x, \omega) = \sum_{j=0}^{n} a_j(\omega) \binom{n}{j}^{1/2} x^j$, Where the coefficients $a_j(\omega)$ are independently normally distributed random variables defined on a probability space (Ω, F, P), $\omega \in \Omega$ is known. In this paper we provide the asymptotic value for the expected number of zeros of the integration of $Q_n(x, \omega)$ with respect to x.

Key words & Phrases: Random algebraic polynomial, Expected number of zeroes

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1. Introduction

Let us consider the random algebraic polynomial as

$$Q_n(x, \omega) = \sum_{j=0}^{n} a_j(\omega) \binom{n}{j}^{1/2} x^j \tag{1.1}$$

Where $a_j(\omega)$, j = 0,1,2,…………,n , $\omega \in \Omega$, is a sequence of independent random variables defined on a probability space (Ω, F, P). Then

$$F_n(t, \omega) = F_n(t) = \int_0^t Q_n(x, \omega)\, dx \tag{1.2}$$

$$= \sum_{j=0}^{n} a_j(\omega) \binom{n}{j}^{1/2} \frac{t^{j+1}}{j+1}$$

represents the area under the curve representing $Q_n(x, \omega)$ between o and $|t|$. Considering the coefficients to be distributed as normal with mean zero and variance unity, Edelman and Kostlan[2] were first to show that the expected number of real zeros of $Q_n(x, \omega)$ in $(-\infty, \infty)$ is asymptotic to $\sqrt{n}$. Later on, the expected number of K-level crossings, the number of maxima and minima or the number of maxima (minima) below (above) a level are studied by Farahmand in[3] and [4]. Further the number of zeros for the area under curve representing a polynomial of the form as in[1] is studied by Farahmand and Jahangiri[5] and showed that it is also equal to $\sqrt{n}$. Now we are motivated to find out the number of zeros for the area under the curve representing the same polynomial, where the coefficients are independently normally distributed random variables with non zero mean, $E(a_j) = \mu\binom{n}{j}^{1/2}$, $\mu \neq 0$ and variance, $V(a_j) = \sigma^2$. The particular choice of polynomial in (1.1) is of special interest for physical properties as described by Ramponi . The analysis employed here may be utilized to find out expected number of zeros for other type of non normal distributions. Hence we have the following theorem[6].

Theorem : If the coefficients a_j of $F_n(t, \omega)$ in (2) are independently normally distributed random variables with expected value $E(a_j) = \mu\binom{n}{j}^{1/2}$, $\mu \neq 0$, and variance, $V(a_j) = \sigma^2$, then the expected number of real zeros of $F_n(t, \omega)$ satisfies

$$EN_F(-\infty, \infty) = \left(\frac{n+1}{2\pi}\right) \log \sqrt{2n}$$

2. *Moments:*

First of all we need to obtain the estimates for the mean of $F_n(t)$, the mean of its derivative with respect to t, $F_n'(t)$, the variance of $F_n(t)$, the variance of its derivative with respect to t, $F_n'(t)$, and the covariance of $F(t)$ and $F_n'(t)$. From the definition of $F_n(t)$ in (1.2) we have.

$$m = E(F_n(t)) = \mu \sum_{j=0}^{n} \binom{n}{j} \frac{t^{j+1}}{j+1} \quad (2.1)$$

$$= \mu \frac{(1+t)^{n+1}}{n+1}$$

$$m' = E(F_n'(t)) = \mu \sum_{j=0}^{n} \binom{n}{j} t^j \quad (2.2)$$

$$= \mu (1+t)^n$$

$$A^2 = V = (F_n(t)) = \sigma^2 \sum_{j=0}^{n} \binom{n}{j} \frac{t^{2j+2}}{(j+1)^2} \quad (2.3)$$

$$\sim \sigma^2 \frac{(1+t^2)^{n+2}}{(n+1)^2}\left[1 + O\left(\frac{1}{n}\right)\right]$$

$$B^2 = V(F_n'(t)) = \sigma^2 \sum_{j=0}^{n} \binom{n}{j} t^{2j} \quad (2.4)$$

$$= \sigma^2 (1+t^2)^n$$

(Using the binomial identity $\sum_{j=0}^{n} \binom{n}{j} t^j = (1+t)^n$)

$$C=\mathrm{Cov}\left(F_n(t),\ Fn'(t)\right)=\sigma^2\sum_{j=0}^{n}\binom{n}{j}\frac{t^{2j+1}}{(j+1)} \qquad (2.5)$$

$$=\sigma^2\frac{(1+t^2)^{n+1}}{(n+1)t}$$

(By integrating the above binomial identity) Then from Cramer and Leadbetter, under the condition that $A^2 B^2 > C^2$, the expected number of real zeros of $F_n(t)$ in the interval (α,β) as,

$$EN_F(\alpha,\beta)=\int\frac{\Delta}{A^2}\varphi\left(\frac{m}{\sigma^2}\right)\left[2\varphi(\eta)+\eta\left(2\Phi(\eta)-1\right)\right]dt \qquad (2.6)$$

Where

$$\Delta^2 = A^2 B^2 - C^2$$

$$\eta=\left(\frac{A^2 m' - Cm}{A\Delta}\right)$$

$$\varphi(t)=\frac{1}{\sqrt{2\pi}}e^{-\left(\frac{t^2}{2}\right)}$$

$$\Phi(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t}\varphi(y)dy$$

Assuming $\mathrm{erf}(x)=\int_0^t e^{-(t^2)}dt$, the equation (2.6) can be written as,

$$EN_F(\alpha,\beta) = I_1(\alpha,\beta) + I_2(\alpha,\beta)$$

Where,

$$I_1(\alpha,\beta)=\int_\alpha^\beta\frac{\Delta}{\pi A^2}e^{\left[-\left(\frac{m^2}{2A^2}\right)+\left(\frac{A^2m'-Cm}{\sqrt{2}A\Delta}\right)^2\right]}dt \qquad (2.7)$$

$$I_2(\alpha,\beta)=\int_\alpha^\beta\left[\frac{\sqrt{2}\left(A^2m'-Cm\right)}{\pi A^3}e^{\left(-\frac{m^2}{2A^2}\right)}erf\left(\frac{A^2m'-Cm}{\sqrt{2}A\Delta}\right)\right]dt \qquad (2.8)$$

3. Proof of the theorem :

Now to estimate $I_2(-\infty,\infty)$, let us assume a new variable 'u' as ,

$$u=\frac{m}{A}$$

$$=\frac{\mu t(1+t)^{n+1}}{\sigma(1+t^2)^{\frac{n+2}{2}}\left[1+O\left(\frac{1}{n}\right)\right]} \qquad (3.1)$$

$$\approx\frac{\mu t(1+t)^{n+1}}{\sigma(1+t^2)^{\frac{n+2}{2}}}$$

for sufficiently large n, $\left[1+O\left(\frac{1}{n}\right)\right]\to 1$

Hence,

$$\frac{du}{dt}=\frac{\mu}{\sigma}(1+t^2)^{-\left(\frac{n+4}{2}\right)}(1+t)^n(1+nt+2t-t^2-nt^2) \qquad (3.2)$$

On Simplification,

$$\left(\frac{\mathrm{A}^2 m' - Cm}{\mathrm{A}^3}\right) = \frac{\mu t(1+t)^n (1+t^2)^{-\left(\frac{n+4}{2}\right)}(1-t)}{\sigma} \quad (3.3)$$

We have , from (2.8) , (3.1) , (3.2) and (3.3)

$$\mathrm{I}_2(-\infty,\infty) \le \int_{-\infty}^{\infty}\left(\frac{\mathrm{A}^2 m' - Cm}{\sqrt{2}\pi\Delta^3}\right) e^{\left(-\frac{m^2}{2\mathrm{A}^2}\right)} dt$$

$$\le \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\left(-\frac{u^2}{2}\right)} du \times \mathrm{O}(1)$$

$$= \mathrm{O}(1) \quad (3.4)$$

Now to estimate $\mathrm{I}_1 \left(-\infty,\infty\right)$, let us assume,

$$\tau(t) = \frac{m^2}{2\mathrm{A}^2} + \left(\frac{\mathrm{A}^2 m' - Cm}{\sqrt{2}\mathrm{A}\Delta}\right)^2$$

$$= \frac{\dfrac{\mu^2 (1+t)^{2n+2}}{(n+1)^2}}{\dfrac{2\sigma^2 (1+t^2)^{n+2}\left[1+\mathrm{O}\left(\frac{1}{n}\right)\right]}{t^2 (n+1)^2}}$$

$$+ \left[\frac{\dfrac{\sigma^2 (1+t^2)^{n+2}\left[1+\mathrm{O}\left(\frac{1}{n}\right)\right]\mu(1+t)^n}{t^2(n+1)^2} - \dfrac{\sigma^2 (1+t^2)^{n+1}\mu(1+t)^{n+1}}{t(n+1)}\dfrac{(1+t)^{2n}(2+t^2-t)}{(n+1)}}{\dfrac{\sqrt{2}\sigma^2 (1+t^2)^{\frac{n+2}{2}}\left[1+\mathrm{O}\left(\frac{1}{n}\right)\right]}{t(n+1)}\dfrac{\sigma^2 (1+t^2)^{n+1}\mathrm{O}\left(\frac{1}{\sqrt{n}}\right)}{t(n+1)}}\right]^2$$

For sufficiently large n, $\mathrm{O}\left(\frac{1}{\sqrt{n}}\right) \to 0$, So second term vanishes

$$\tau(t) = \frac{\mu^2}{2\sigma^2}\frac{t^2 (1+t)^{2n+2}}{(1+t^2)^{n+2}}$$

$$= \frac{\mu^2}{2\sigma^2}\left(\frac{t^2}{1+t^2}\right)\left(\frac{(1+t)^2}{(1+t^2)}\right)^{(n+1)} \quad (3.5)$$

Now $\Delta^2 = \mathrm{A}^2 \mathrm{B}^2 - \mathrm{C}^2$

$$= \frac{\sigma^4 (1+t^2)^{2n+2}}{t^2 (n+1)^2}\mathrm{O}\left(\frac{1}{n}\right)$$

So ,

$$\Delta = \frac{\sigma^2 (1+t^2)^{n+1}}{t\ (n+1)}\ \mathrm{O}\left(\frac{1}{\sqrt{n}}\right) \quad (3.6)$$

So that,

$$\frac{\Delta}{A^2} = \frac{\sqrt{n}t}{(1+t^2)} \qquad (3.7)$$

Now Consider $0 < t < 1$, the integral $I_1(0,1)$ can be written as

$$I_1(0,1) = I_1(0,\varepsilon) + I_1(\varepsilon,1) \qquad (3.8)$$

Where $\varepsilon = \frac{1}{\sqrt{n}} \to 0$, as $n \to \infty$

Then,

$$I_1(0,\varepsilon) = \int_0^{\varepsilon} \frac{\Delta}{\pi A^2} \exp\,(-\tau(t))\ dt$$

$$\leq \frac{\sqrt{n}}{\pi} \int_0^{\varepsilon} \frac{t}{(1+t^2)} dt \text{ (Where } \tau(t) \to 0, \text{ as } n \to \infty)$$

$$= \frac{\sqrt{n}}{\pi}\left[\log(1+t^2)\right]_0^{\varepsilon}$$

$$= \frac{\sqrt{n}}{\pi} \log\left(1+\frac{1}{n}\right) \text{(Choosing } \varepsilon = \frac{1}{\sqrt{n}})$$

$\to 0$, as $n \to \infty$

So,

$$I_1(0,\varepsilon) \to 0 \text{ as } n \to \infty \qquad (3.9)$$

Next consider $\varepsilon < t < 1$,

So, $$I_1(\varepsilon,1) = \int_{\varepsilon}^{1} \frac{\Delta}{\pi A^2} e^{(-\tau(t))}\, dt$$

$$= \frac{\sqrt{n}}{\pi} \int_{\varepsilon}^{1} \frac{t}{(1+t^2)} e^{(-\tau(t))}\, dt$$

$\to 0$, as $\tau(t) \to \infty$, where $n \to \infty$

Therefore ,

$$I_1(\varepsilon,1) \to 0 \text{ as } n \to \infty \qquad (3.10)$$

Hence from (3.9) and (3.10), we obtain

$$I_1(0,1) \to 0 \text{ as } n \to \infty \qquad (3.11)$$

Then consider $-1 < t < 0$, the integral, $I_1(-1,0)$ can be written as

$$I_1(-1,0) = I_1(-1,-\varepsilon) + I_1(-\varepsilon,0) \qquad (3.12)$$

So

$$I_1(-\varepsilon,0) = \int_{-\varepsilon}^{0} \frac{\Delta}{\pi A^2} e^{(-\tau(t))}\, dt$$

$$\leq \frac{\sqrt{n}}{\pi} \int_{-\varepsilon}^{0} \frac{t}{(1+t^2)} dt \text{ (Where } \tau(t) \to 0, \text{ as } n \to \infty)$$

$$= \frac{\sqrt{n}}{\pi}\left[\log(1+t^2)\right]_{-\varepsilon}^{0}$$

$$= \frac{-\sqrt{n}}{\pi} \log(1+\varepsilon^2)$$

$$= \frac{-\sqrt{n}}{\pi} \log\left(1+\frac{1}{n}\right) \text{(Choosing } \varepsilon = \frac{1}{\sqrt{n}})$$

$\to 0$ as $n \to \infty$

So,

$$I_1(-\varepsilon,0) \to 0 \text{ as } n \to \infty \qquad (3.13)$$

Next consider $-1<t<-\varepsilon$, We have,

$$\tau(t)=\frac{\mu^2}{4\sigma^2}\left(\frac{t^2}{1+t^2}\right)\left(\frac{(1+t)^2}{(1+t^2)}\right)^{(n+1)}$$

Putting t=-y, so that $\varepsilon<y<1$, then $\tau(t)$ becomes $\tau_1(y)$ as

$$\tau_1(y)=\frac{\mu^2}{4\sigma^2}\left(\frac{y^2}{1+y^2}\right)\left(\frac{(1-y)^2}{(1+y^2)}\right)^{(n+1)} \quad (3.14)$$

Where

$$\tau_1(y)\to 0 \text{ as } n\to\infty$$

So for

$-1<t<-\varepsilon$

$$I_1(-1,-\varepsilon)=\int_{-1}^{-\varepsilon}\frac{\Delta}{\pi A^2}\,e^{(-\tau(t))}\,dt$$

$$=\frac{\sqrt{n}}{\pi}\int_{-1}^{-\varepsilon}\frac{t}{(1+t^2)}\,e^{(-\tau(t))}\,dt$$

For $\varepsilon<y<1$

Now $I_1(\varepsilon,1)=\frac{\sqrt{n}}{\pi}\int_{\varepsilon}^{1}\frac{y}{(1+y^2)}\,e^{(-\tau_1(y))}\,dt$

$$=\frac{\sqrt{n}}{\pi}\int_{\varepsilon}^{1}\frac{y}{(1+y^2)}\,dy \text{ (For sufficiently large n)}$$

$$=\frac{\sqrt{n}}{\pi}\left[\log(1+y^2)\right]_{\varepsilon}^{1}$$

$$=\frac{\sqrt{n}}{\pi}\log\left(\frac{2n}{n+1}\right)^{\frac{1}{2}} \quad (3.15)$$

Then from (3.13) & (3.15) it follows that

$$I_1(-1,0)=\frac{\sqrt{n}}{\pi}\log\left(\frac{2n}{n+1}\right)^{\frac{1}{2}} \quad (3.16)$$

Consider $1<t<\infty$, put $t=\frac{1}{y}$, So that $0<y<1$

For $1<t<\infty$, $I_1(1,\infty)=I_1(0,1)$, For $0<y<1$

$= I_1(0,\varepsilon) + I_1(\varepsilon,1)$

We have

$$\tau(t)=\frac{\mu^2}{4\sigma^2}\left(\frac{t^2}{1+t^2}\right)\left(\frac{(1+t)^2}{(1+t^2)}\right)^{(n+1)}$$

For $t=\frac{1}{y}$, $\tau(t)$ becomes $\tau_2(y)$ as

$$\tau_2(y)=\frac{\mu^2}{4\sigma^2}\frac{1}{(1+y^2)}\left(\frac{(1+y)^2}{(1+y^2)}\right)^{(n+1)} \quad (3.17)$$

For $1<t<\infty$,

$$I_1(1,\infty)=\int_{1}^{\infty}\frac{\Delta}{\pi A^2}\,e^{(-\tau(t))}\,dt$$

$$= \frac{\sqrt{n}}{\pi} \int_{1}^{\infty} \frac{t}{(1+t^2)} \, e^{(-\tau(t))} \, dt$$

For $0 < y < 1$

Now $I_1(0,1) = I_1(0, \varepsilon) + I_1(\varepsilon, 1)$

$$I_1(0,\varepsilon) = \frac{-(n+1)}{2\pi} \int_{0}^{\varepsilon} \frac{1}{y(1+y^2)} \, e^{\left(-\tau_2(y)\right)} \, dy$$

$$= \frac{(n+1)}{2\pi} \log \sqrt{n+1} \qquad (3.18)$$

$$I_1(\varepsilon,1) = \frac{-(n+1)}{2\pi} \int_{\varepsilon}^{1} \frac{1}{y(1+y^2)} \, e^{\left(-\tau_2(y)\right)} \, dy$$

$\rightarrow 0$ as $\tau_2(y) \rightarrow \infty 0$ as For sufficiently large n (3.19)

Then from (3.18) and (3.19), it follows that

$$I_1(1,\infty) = \frac{\sqrt{n}}{\pi} \log \sqrt{n+1} \qquad (3.20)$$

Consider $-\infty < t < -1$, put y = $\frac{1}{t}$, So that -1<y<0,

The Integral $I_1(-\infty,-1)$ for $-\infty < t < -1$ can be written as $I_1(-1,0)$ for -1<y<0

For $-\infty < t < -1$

$$I_1(-\infty,-1) = \frac{\sqrt{n}}{\pi} \int_{-\infty}^{-1} \frac{t}{(1+t^2)} \, e^{(-\tau(t))} \, dt$$

For -1<y<0

$$I_1(-1,0) = \frac{-\sqrt{n}}{\pi} \int_{-1}^{0} \frac{1}{y(1+y^2)} \, e^{(-\tau_2(y))} \, dy$$

Using (3.17)

$$= \frac{-\sqrt{n}}{\pi} \int_{-1}^{0} \frac{1}{y(1+y^2)} \, dy$$ = dy (as $\tau_2(y) \rightarrow 0$ as n $\rightarrow \infty$)

$$= \frac{-\sqrt{n}}{\pi} \int_{\frac{3\pi}{4}}^{0} \frac{1}{\tan\theta} \, d\theta$$

(Putting y=tanθ $\Rightarrow dy = \sec^2\theta \, d\theta$

$$= \frac{-\sqrt{n}}{\pi} \left[\log \sin 0\right]_{\frac{3\pi}{4}}^{0}$$

$$= \frac{-\sqrt{n}}{\pi} \log \frac{1}{\sqrt{2}}$$

= 0 (Assuming log 0=0) (3.21)

Therefore from (3.11), (3.16), (3.20) and (3.21) it follows that

$$I_1(-\infty,\infty) = \frac{-\sqrt{n}}{\pi} \log\left(\frac{2n}{n+1}\right)^{\frac{1}{2}}$$

Hence

$EN_F(-\infty,\infty) = I_1(-\infty,\infty) + I_2(-\infty,\infty)$

$$\sim \frac{\sqrt{n}}{\pi} \log\left(\frac{2n}{n+1}\right)^{\frac{1}{2}}$$

Which proves the theorem.

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