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# Integrability Conditions of a Generalised almost contact manifold

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**Abstract**

In this Paper, we have obtained the integrability conditions for a generalized almost contact manifold some related result have also been discussed.

## 1. Introduction

An odd dimensional Riemannian manifold ($V_n$, g) is said to be Generalised almost contact manifold[1-9] if there exists a tensor φ of the type (1,1) and a global vector field ξ and a 1-form η satisfying the following equations:

$$\phi^2 X = a^2 X + \eta(X)\xi, \tag{1.1)a}$$

$$\eta(\phi X) = 0, \tag{1.1)b}$$

$$\eta(\xi) = -a^2, \tag{1.1)c}$$

$$\phi(\xi) = 0, \tag{1.1)d}$$

$$\eta(X) = g(X,\xi), \tag{1.1)e}$$

$$g(\varphi X.\varphi Y) = -a^2 g(X,Y) - \eta(X)\eta(Y). \tag{1.1)f}$$

Where X, Y $\in V_n$ '$a$' be a complex number and g be the metric of $V_n$.

From the above definition it is clear that almost contact manifold is a particular case of a Generalised almost contact manifold for $a^2 = -1$.

The Nijenhuis[10-11] tensor with respect to the structure φ on a generalized almost contact manifold is given by

$$N(X,Y) = [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + \phi^2[X,Y]$$

*2. Integrability Condition:*

*Theorem(2.1). If* α be an eigen value of φ then

$$\alpha(\alpha^2 - a^2)^m = 0.$$

*Proof:* If P is an eigen vector of φ

corresponding to the eigen value α then

$\phi P = \alpha P$,

or

$$\overline{P} = \alpha P \cdot \quad (2.1)$$

Applying again φ in equation (2.1) we get

$$\alpha \overline{P} = \alpha \phi P,$$

$$\phi^2 P = \alpha^2 P. \quad (2.2)$$

Applying equation (1.1)a in equation (2.2) we get

$$a^2 P + \eta(P)\xi = \alpha^2 P,$$

$$(\alpha^2 - a^2)P = \eta(P)\xi. \quad (2.3)$$

Two cases arises:

*Case 1*: If P and ξ are linearly dependent then $P = K\xi$, then 0 is an eigen value, the corresponding eigen vector being ξ. ξ is the only vector such that $\overline{\xi} = 0$, therefore the eigen value 0 is of multiplicity 1.

*Case 2:* If {P, ξ) are linearly independent then $\alpha^2 - a^2 = 0$. Hence eigen values a,-a occur in pairs[2-5]. Since rank (φ) =2m, then multiplicity of a or –a is m. There is a pencil of eigen vectors corresponding to the eigen value 'a' and a pencil of complex conjugate eigen vectors corresponding to the eigen value '-a'. If P is an eigen vector corresponding to a or –a, $\eta(P) = 0$.

*Theorem (2.2):* The necessary and sufficient condition that $V_n$ (*n =2m+1*) be an Generalised almost contact manifold is that it contains a tangent bundle $\pi_m$ of dimension m, a tangent bundle $\overline{\pi_m}$ conjugate to $\pi_m$ and a real line $\pi_1$ such that

$$\pi_m \cap \overline{\pi_m} = \phi,\ \pi_m \cap \pi_1 = \phi,\ \overline{\pi_m} \cap \pi_1 = \phi,$$

and

$\pi_1 \cup \pi_m \cup \overline{\pi_m} = a$ tangent bundle of dimension $n = 2m+1$, projections *l*, *m*, *n* on $\pi_m$, $\overline{\pi_m}$, $\pi_1$ being given by

$$2la^2 = \phi^2 + a\phi, \quad (2.4)a$$

$$2ma^2 = \phi^2 - a\phi, \quad (2.4)b$$

$$n = \phi^2 - a^2 = \eta \otimes \xi. \quad (2.4)c$$

*Necessary condition:*

Let $V_n$ be an Generalised almost contact manifold with Generalised almost contact structure {φ, ξ, η} corresponding to the eigen value a , Let $P_x$, x = 1,2,3.........*m* be linearly independent eigen vectors. Let $Q_x$ be the complex conjugate to $P_x$, then[6-11]

$$b^x P_x = 0 \Rightarrow b^x = 0 \ \forall x,$$

$$c^x P_x = 0 \Rightarrow c^x = 0 \ \forall x,$$

now

$$d^x P_x + e^x Q_x + h\xi = 0 \Rightarrow d^x \overline{P_x} + e^x \overline{Q_x} = 0$$

$$\Rightarrow ad^x P_x - ae^x Q_x = 0$$

$$\Rightarrow d^x P_x - e^x Q_x = 0.$$

All these equations imply

$$h\xi = d^x P_x = e^x Q_x = 0,$$

or

$h = d^x = e^x = 0 \cdot \quad \forall x$

Thus $\{P_x, Q_x, \xi\}$ is linearly independent set. From (2.4)a, (2.4)b, (2.4)c we have

$lP_x = P_x$, (2.5)a $lQ_x = 0$, (2.5)b $l\xi = 0$; (2.6)c

$mP_x = 0$, (2.6)a $mQ_x = Q_x$, (2.6)b $m\xi = 0$; (2.6)c

$nP_x = 0$, (2.7)a $nQ_x = 0$, (2.7)b $n\xi = \xi$. (2.7)c

Thus we prove that on generalized almost contact manifold $V_{2m+1}$ there is a tangent bundle $\pi_m$ of dimension *m*, a tangent bundle $\overline{\pi_m}$ complex conjugate to $\pi_m$ and a real line such that

$\pi_m \cap \overline{\pi_m} = \phi, \ \pi_m \cap \pi_1 = \phi, \ \overline{\pi_m} \cap \pi_1 = \phi,$

and

$\pi_1 \cup \pi_m \cup \overline{\pi_m} = a$ tangent bundle of dimension $n = 2m+1$, projections on $\pi_m$, $\overline{\pi_m}$, $\pi_1$ being *l*, *m* and *n* respectively.

*Sufficient Condition:*

Suppose that there is a tangent bundle $\pi_m$ of dimension *m*, a tangent bundle $\overline{\pi_m}$ complex conjugate to $\pi_m$ and a real line $\pi_1$ such that

$\pi_m \cap \overline{\pi_m} = \phi, \ \pi_m \cap \pi_1 = \phi, \ \overline{\pi_m} \cap \pi_1 = \phi,$

and

$\pi_1 \cup \pi_m \cup \overline{\pi_m} = a$ tangent bundle of dimension $n = 2m+1 \cdot$

Let $P_x$ be *m* linearly independent vectors in $\pi_m$ and $Q_x$ complex conjugate to $P_x$ be *m* linearly independent vectors in $\overline{\pi_m}$ and ξ be a vector in $\pi_1$. Let $\{P_x, Q_x, \xi\}$ span a tangent bundle dimension $2m+1$. Then $\{P_x, Q_x, \xi\}$ is a linearly independent set. Let us define the inverse set $\{p^x, q^x, \eta\}$ such that

$$I_n = p^x \otimes P_x + q^x \otimes Q_x - \frac{\eta \otimes \xi}{a^2}. \qquad (2.8)$$

Let us put

$$\phi = a\{p^x \otimes P_x - q^x \otimes Q_x\}, \qquad (2.9)a$$

then

$$\phi^2 = a^2\{p^x \otimes P_x + q^x \otimes Q_x\}. \qquad (2.9)b$$

Using equation (2.8) in equation (2.9)b, we get

$$\phi^2 = a^2 I_n + \eta \otimes \xi . \qquad (2.10)$$

The equation (2.10 ) shows that the manifold admit an generalized almost contact structure {φ, ξ, η}. Hence the condition is sufficient also[1-7].

*Corollary (2.1):* We have

$$l = p^x \otimes P_x \qquad (2.11)a$$

$$m = q^x \otimes Q_x \qquad (2.11)b$$

$$n = \eta \otimes \xi \qquad (2.11)c$$

*Proof:* From equations (2.4)a and (2.4)b, we have

$$\phi = a(l - m), \qquad (2.12)a$$

$$\phi^2 = a^2(l + m). \qquad (2.12)b$$

From (2.9)a and (2.9)b , we have

$$\phi = a\{p^x \otimes P_x - q^x \otimes Q_x\}, \qquad (2.13)a$$

$$\phi^2 = a^2(p^x \otimes P_x + q^x \otimes Q_x\}. \qquad (2.13)b$$

Comparing equations (2.12) and (2.13), we get the equations (2.11).

*Corollary (2.2):* We have

$$lm = ml = \ln = nl = mn = nm = 0\,, \qquad (2.14)$$

and

$$l^2 = l\,, \qquad m^2 = m\,, \qquad n^2 = n\,. \qquad (2.15)$$

*Proof:* From equations (2.5)b and (2.11)b, we get

$$lm = q^x \otimes lQ_x = 0\,.$$

From equations (2.5)a and (2.11)a , we have

$$l^2 = p^x \otimes lP_x = p^x \otimes P_x = l\,.$$

Similarly we can prove other relations.

*Lemma (2.1):* $(dl)(nX, nY) = 0\,,$ (2.16)a

$$(dp^x)(nX, nY) = 0\,. \qquad (2.16)b$$

*Proof:* (2.11)c yields (2.16)a . For

$$(dl)(\eta(X)\xi, \eta(Y)\xi) = 0\,.$$

Since d is skew symmetric in both the slots. Proof of (2.16)b follows the same pattern.

*Lemma (2.2):* $2(dm)(lX, lY) = -\frac{1}{a^2}\overline{\overline{[lX.lY]}} + \frac{1}{a}\overline{[lX, lY]},$

(2.16)a

$$8(dm)(lX, lY) = -\frac{1}{a^6}\overline{\overline{[\bar{\bar{X}}, \bar{\bar{Y}}]}} + \frac{1}{a^5}\overline{N(\bar{X}, \bar{Y})}$$

$$+\frac{1}{a^4}\overline{\overline{N(\bar{X}, \bar{Y})}} - \frac{1}{a^4}\overline{\overline{[\bar{\bar{X}}, \bar{\bar{Y}}]}} + \frac{1}{a^3}\overline{[\bar{X}, \bar{Y}]}$$

$$+\frac{1}{a^5}\overline{[\bar{X}, \bar{Y}]}\,. \qquad (2.16)b$$

*Proof:* From equation (2.14) , we have

$$2(dm)(lX, lY) = -2m[lX, lY]\,,$$

$$= -\frac{1}{a^2}\overline{\overline{[lX, lY]}} + \frac{1}{a}\overline{[lX, lY]}\,.$$

Now

$$8(dm)(lX, lY) = -\frac{1}{a^2}\overline{\overline{[2lX, 2lY]}} + \frac{1}{a}\overline{[2lX, 2lY]}\,,$$

$$= -\frac{1}{a^2}\{\frac{1}{a^4}\overline{\overline{[\bar{\bar{X}}, \bar{\bar{Y}}]}} + \frac{1}{a^3}\overline{\overline{[\bar{\bar{X}}, \bar{Y}]}} + \frac{1}{a^3}\overline{\overline{[\bar{X}, \bar{\bar{Y}}]}} + \frac{1}{a^2}\overline{\overline{[\bar{X}, \bar{Y}]}}\}$$

$$+\frac{1}{a}\{\frac{1}{a^4}\overline{[\bar{\bar{X}}, \bar{\bar{Y}}]} + \frac{1}{a^3}\overline{[\bar{\bar{X}}, \bar{Y}]} + \frac{1}{a^3}\overline{[\bar{X}, \bar{\bar{Y}}]} + \frac{1}{a^2}\overline{[\bar{X}, \bar{Y}]}\}\,,$$

$$= -\frac{1}{a^6}\overline{\overline{[\bar{\bar{X}}, \bar{\bar{Y}}]}} - \frac{1}{a^5}\overline{\overline{[\bar{\bar{X}}, \bar{Y}]}} - \frac{1}{a^5}\overline{\overline{[\bar{X}, \bar{\bar{Y}}]}} - \frac{1}{a^4}\overline{\overline{[\bar{X}, \bar{Y}]}} + \frac{1}{a^5}\overline{[\bar{\bar{X}}, \bar{\bar{Y}}]}$$

$$+\frac{1}{a^4}\overline{[\bar{\bar{X}}, \bar{Y}]} + \frac{1}{a^4}\overline{[\bar{X}, \bar{\bar{Y}}]} + \frac{1}{a^3}\overline{[\bar{X}, \bar{Y}]}\,,$$

$$= -\frac{1}{a^6}\overline{\overline{[\bar{\bar{X}}, \bar{\bar{Y}}]}} + \frac{1}{a^5}\{\overline{[\bar{\bar{X}}, \bar{\bar{Y}}]} - \overline{\overline{[\bar{\bar{X}}, \bar{Y}]}} - \overline{\overline{[\bar{X}, \bar{\bar{Y}}]}} - \overline{[\bar{X}, \bar{Y}]}\} + \frac{1}{a^5}\overline{[\bar{X}, \bar{Y}]}$$

$$+\frac{1}{a^4}\{\overline{\overline{[\bar{\bar{X}},\bar{\bar{Y}}]}}+\overline{[\bar{\bar{X}},\bar{Y}]}+\overline{[\bar{X},\bar{\bar{Y}}]}-\overline{\overline{[\bar{X},\bar{Y}]}}\}-\frac{1}{a^4}\overline{\overline{[\bar{\bar{X}},\bar{\bar{Y}}]}}+\frac{1}{a^3}\overline{[\bar{X},\bar{Y}]},$$

$$8(dm)(lX,lY)=-\frac{1}{a^6}\overline{\overline{[\bar{\bar{X}},\bar{\bar{Y}}]}}+\frac{1}{a^5}\overline{N(\bar{X},\bar{Y})}+\frac{1}{a^4}\overline{\overline{N(\bar{X},\bar{Y})}}-\frac{1}{a^4}\overline{\overline{[\bar{\bar{X}},\bar{\bar{Y}}]}}+\frac{1}{a^3}\overline{[\bar{X},\bar{Y}]}+\frac{1}{a^5}\overline{[\bar{X},\bar{Y}]}.$$

*Lemma (2.3):* $dn(lX,lY)=-\eta[lX,lY]\xi$, (2.17)

$$4dn(lX,lY)=\frac{1}{a^3}\overline{N(\bar{X},\bar{Y})}-\frac{1}{a^2}\overline{\overline{N(X,Y)}}+\frac{1}{a^2}N(\bar{X},\bar{Y})+\frac{1}{a}\overline{N(\bar{X},\bar{Y})}-\frac{1}{a^4}\overline{\overline{[\bar{\bar{X}},\bar{\bar{Y}}]}}-\frac{1}{a^3}\overline{[\bar{\bar{X}},\bar{\bar{Y}}]}$$

$$+\frac{1}{a^3}\overline{[\bar{X},\bar{Y}]}+\frac{1}{a^2}[\bar{X},\bar{Y}]+\frac{1}{a^2}[X,Y]-\frac{1}{a}\overline{[\bar{\bar{X}},\bar{\bar{Y}}]}+\frac{1}{a}\overline{[\bar{X},\bar{Y}]}+[\bar{X},\bar{Y}]. \quad (2.17)\text{b}$$

*Proof:* From (2.14) , we get

$$dn(lX,lY)=-\eta[lX,lY]\xi\,.$$

Now

$$4(dn)(lX,lY)=-4n[lX,lY],$$

$$=-4\overline{\overline{[lX,lY]}}+4a^2[lX,lY],$$

$$=-\overline{\overline{[\frac{1}{a^2}\bar{\bar{X}}+\frac{1}{a}\bar{X},\frac{1}{a^2}\bar{\bar{Y}}+\frac{1}{a}\bar{Y}]}}+a^2[\frac{1}{a^2}\bar{\bar{X}}+\frac{1}{a}\bar{X},\frac{1}{a^2}\bar{\bar{Y}}+\frac{1}{a}\bar{Y}],$$

$$=-\frac{1}{a^4}\overline{\overline{[\bar{\bar{X}},\bar{\bar{Y}}]}}-\frac{1}{a^3}\overline{\overline{[\bar{\bar{X}},\bar{Y}]}}-\frac{1}{a^3}\overline{\overline{[\bar{X},\bar{\bar{Y}}]}}-\frac{1}{a^2}\overline{\overline{[\bar{X},\bar{Y}]}}+\frac{1}{a^2}[\bar{\bar{X}},\bar{\bar{Y}}]+\frac{1}{a}[\bar{\bar{X}},\bar{Y}]+\frac{1}{a}[\bar{X},\bar{\bar{Y}}]+[\bar{X},\bar{Y}],$$

$$=\frac{1}{a^3}\overline{N(\bar{X},\bar{Y})}-\frac{1}{a^2}\overline{\overline{N(X,Y)}}+\frac{1}{a^2}N(\bar{X},\bar{Y})+\frac{1}{a}\overline{N(\bar{X},\bar{Y})}-\frac{1}{a^4}\overline{\overline{[\bar{\bar{X}},\bar{\bar{Y}}]}}-\frac{1}{a^3}\overline{[\bar{\bar{X}},\bar{\bar{Y}}]}+\frac{1}{a^3}\overline{[\bar{X},\bar{Y}]}$$

$$+\frac{1}{a^2}[\bar{X},\bar{Y}]+\frac{1}{a^2}[X,Y]-\frac{1}{a}\overline{[\bar{\bar{X}},\bar{\bar{Y}}]}+\frac{1}{a}\overline{[\bar{X},\bar{Y}]}+[\bar{X},\bar{Y}].$$

*Theorem (2.2):* The real line is integrable.
*Proof*: From equation (2.8) , we have

$$I_n=l+m-\frac{n}{a^2}\,.$$

In order to that $\pi_1$ is integrable, it is necessary and sufficient that $l=0$ and $m=0$ be integrable that is

$$dl(X,Y)=0\Rightarrow dl(a^2X,a^2Y)=0$$

$$\Rightarrow dl(nX,nY)=0\,.$$

But from lemma (2.1), these equations are identically satisfied, hence we have the statement[8-11].

*Theorem(2.3):* In order that $\pi_m$ and $\overline{\pi_m}$ are integrable, it is necessary and sufficient that

$$\overline{\overline{\frac{1}{a^{2}}[lX,lY]}}=\frac{1}{a}\overline{[lX,lY]}, \tag{2.18)a}$$

$$\frac{1}{a^{5}}\overline{N(\bar{X},\bar{Y})}+\frac{1}{a^{4}}\overline{\overline{N(\bar{X},\bar{Y})}}+\frac{1}{a^{3}}\overline{[\bar{X},\bar{Y}]}+\frac{1}{a^{5}}\overline{[\bar{X},\bar{Y}]}=\frac{1}{a^{6}}\overline{\overline{[\bar{\bar{X}},\bar{\bar{Y}}]}}+\frac{1}{a^{4}}\overline{\overline{[\bar{\bar{X}},\bar{\bar{Y}}]}}\ , \tag{2.18)b}$$

$$\eta[lX,lY]\xi=0, \tag{2.18)c}$$

$$\frac{1}{a^{3}}\overline{N(\bar{X},\bar{Y})}+\frac{1}{a}\overline{N(\bar{X},\bar{Y})}+\frac{1}{a^{2}}N(\bar{X},\bar{Y})+\frac{1}{a^{3}}\overline{[\bar{X},\bar{Y}]}+\frac{1}{a^{2}}[\bar{X},\bar{Y}]+\frac{1}{a^{2}}[X,Y]+\frac{1}{a}\overline{[\bar{X},\bar{Y}]}+[\bar{X},\bar{Y}]$$
$$=\frac{1}{a^{2}}\overline{\overline{N(X,Y)}}+\frac{1}{a^{4}}\overline{\overline{[\bar{\bar{X}},\bar{\bar{Y}}]}}+\frac{1}{a^{3}}\overline{[\bar{\bar{X}},\bar{\bar{Y}}]}+\frac{1}{a}[\bar{\bar{X}},\bar{\bar{Y}}]. \tag{2.18)d}$$

*Proof:* In order to that $\pi_m$ is completely integrable, it is necessary and sufficient that $m$=0, $n$=0 be completely integrable, that is

$$dm(X,Y)=0, \tag{2.19)a}$$
$$dn(X,Y)=0. \tag{2.19)b}$$

Now from equation (2.8) we have $l=I_n$

So
$$dm(lX,lY)=0, \tag{2.20)a}$$
$$dn(lX,lY)=0. \tag{2.20)b}$$

From equation (2.19)a , we get

$$dm(lX,lY)=0,$$
$$8dm(lX,lY)=0.$$

From lemma (2.2) , we have

$$\frac{1}{a^{2}}\overline{\overline{[lX,lY]}}=\frac{1}{a}\overline{[lX,lY]},$$

and

$$\frac{1}{a^5}\overline{N(\overline{X},\overline{Y})}+\frac{1}{a^4}\overline{\overline{N(\overline{X},\overline{Y})}}+\frac{1}{a^3}\overline{[\overline{X},\overline{Y}]}+\frac{1}{a^5}\overline{[\overline{X},\overline{Y}]}=\frac{1}{a^6}\overline{\overline{[\overline{\overline{X}},\overline{\overline{Y}}]}}+\frac{1}{a^4}\overline{\overline{[\overline{\overline{X}},\overline{\overline{Y}}]}}\ .$$

Now from equation (2.19)b , we have

$$dn(X,Y)=0\,.$$

From equation (2.8) we have $l=I_n$, so

$$dn(lX,lY)=0\,.$$

From lemma (2.3) , we have

$$dn(lX,lY)=-\eta[lX,lY]\xi=0.$$

And

$$4dn(lX,lY)=0$$

Now from lemma (2.3) , we get

$$\frac{1}{a^3}\overline{N(\overline{X},\overline{Y})}+\frac{1}{a}\overline{N(\overline{X},\overline{Y})}+\frac{1}{a^2}N(\overline{X},\overline{Y})+\frac{1}{a^3}\overline{[\overline{X},\overline{Y}]}+\frac{1}{a^2}[\overline{X},\overline{Y}]$$

$$+\frac{1}{a^2}[X,Y]+\frac{1}{a}\overline{[\overline{X},\overline{Y}]}+[\overline{X},\overline{Y}]$$

$$=\frac{1}{a^2}\overline{\overline{N(X,Y)}}+\frac{1}{a^4}\overline{\overline{[\overline{\overline{X}},\overline{\overline{Y}}]}}+\frac{1}{a^3}\overline{[\overline{\overline{X}},\overline{\overline{Y}}]}+\frac{1}{a}[\overline{\overline{X}},\overline{\overline{Y}}].$$

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