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On the Topology of Almost Kaehlerian Manifolds

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Abstract

Tachibana (1965) has studied on harmonic tensor in compact Sasakian spaces. Chern (1966) has studied the geometry of G-structures. After than, Blair (1967) has studied the theory of Quasi-Sasakian structures. In this paper, we have studied on the topology of Almost Kaehlerian manifold and several theorems have been established. The notation and terminology in this paper will be the same as that employed in Blair (1967).

Key words & Phrases: Kaehler and Riemannian geometry, Topology, Compact, Almost Kaehlerian Manifolds.

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1. Introduction

Chern[2] asked for the conditions on G-structures, both local and global, on a $\boldsymbol{C}^\infty$ manifold in order that a linear differential form η exist such that

$$\boldsymbol{\eta} \wedge (\boldsymbol{d\eta})^p \neq 0$$

for a given value of p. The form η defines a differential system and it is important to study the local and global properties of its integral manifolds. To this end, the notion of a quasi-Sasakian structure on an almost contact metric developed and topological properties are considered and it is shown that both compact Sasakian and cosymplectic manifolds have global properties similar to compact Kaehler manifolds.

We have define two operators $\boldsymbol{L}$ and $\mathbf{A}$, dual to each other, on a quasi-Kaehlerian manifold by $\boldsymbol{L} = \epsilon(\emptyset)$ and $\mathbf{A} = \iota(\phi)$ where ϵ and ι are respectively the exterior and interior product operators. We say that a p-form α $(p \geq 2)$ is eflective if A $\alpha = 0$. Since $\iota(\phi) = *\epsilon(\emptyset)*$ where * is the Hodge star isomorphism, $\boldsymbol{A} = *\boldsymbol{L}*$.

An orthonormal basis of $\mathbf{O}^{2n+1}$ on an almost contact metric manifold M^{2n+1} of the form $\{\xi, X_i, X_{i^*} = \emptyset X_i\}$, i = 1, . . ., n, is called a $\emptyset$ —basis. It is well known that such a basis always exists. For, let V = {X $\in M_m$ | $g(X, \xi) = 0$}. Blair[1], shows that $\emptyset$ | v is an almost complex structure on V and g/v is a Hermitian metric. If an orthonormal basis of V of the form $\{X_i, (\emptyset/v)X_i\}$, i=1,…..,n is then chosen, we obtain a $\emptyset$ —basis of M_m.

In terms of a $\emptyset$ —basis $\{\xi, X_i, X_{i^*}\}$ with dual basis $\{\eta, \omega_i, \omega_{i^*}\}$ we have $\phi = \sum_i \omega_i \wedge \omega_{i^*}$, $A = \sum_i \iota(\omega_{i^*})\, \iota(\omega_i)$.

Definition(1.1): On a quasi-Kaehlerian manifold M^{2n+1} the operators L and **A** satisfy (AL–LA)α = $(n-p)\alpha$ for any p-form[4-5].

Definition (1.2): In a compact (2n+1) -dimensional quasi-Kaehlerian manifold of rank 2n+1 or 1 the operator C sends harmonic p-forms into harmonic p-forms for p ≤ *n* in the Kaehlerian case and for p = 1,……,2n in the cosymplectic case given by Tachibana[6].

2. Topology of Kaehlerian Manifolds:

We have the following-

Theorem (2.1): The p-th betti number of a compact Kaehlerian manifold M^{2n+1} is even if p is odd and p ≤ n. For p ≥ **n** + 1, $\mathbf{B_p}$ is even if p is even.

Proof: The second statement follows from the first by Poincare duality. So let a be a harmonic p-form with p≤*n;* we shall show that α and C_α are independent, that is, C_α=/=λ_α By using Blair[1]

$$C^2 \alpha(X_1,\ldots\ldots,X_p) == \alpha(\emptyset^2 X_1,\ldots\ldots\ldots, X_p)$$

$$== \alpha[-X_1+\eta(X_1)\xi,\ldots,-X_p+\eta(X_p)\xi]$$

$$== (-1)^p \alpha(X_1,\ldots\ldots,X_p)$$

Since $\iota(\xi)\alpha = 0$. Hence if $C_\alpha = 0$, α must also vanish. Suppose now that Cα = λα. Then $C^2 \alpha = \lambda C\alpha = \lambda^2\alpha$. But $C^2 \alpha = (-1)^p \alpha$, so if p is odd, $\lambda^2\alpha = -\alpha$, that is α = 0.

Theorem (2.2): There are no covariant constant p-forms on a compact Kaehlerian manifold M^{2n+1} for 1≤*p*≤ 2n.

Proof: Let α be a covariant constant p-form with 1≤ *p* ≤ n and let $X, Y_2, \ldots, Y_p \in \mathbf{O}^{2n+1}$. Then since $\iota(\xi)\alpha = 0$ and $\nabla_x \xi = -\frac{1}{2}\emptyset X$, By using Blair[1]. We have

$$0 = [\nabla_{\emptyset x}\{\iota(\xi)\alpha\}](Y_2,\ldots,Y_p) =$$

$$\{\iota(\nabla_{\emptyset x}\xi)\alpha\}(Y_2,\ldots,Y_p)$$

$$= -\frac{1}{2}\alpha(\emptyset^2 X, Y_2,\ldots,Y_p)$$

$$= \frac{1}{2}\alpha(X, Y_2,\ldots,Y_p).$$

Thus α = 0. The same is true for forms of degree p, *n* + 1 ≤ *p* ≤ 2n, since *α is covariant constant whenever α is, and * is an isomorphism.

Theorem (2.3): If the Ricci curvature of a compact Kaehlerian manifold is positive semi-definite, $B_1 = 0$.

Proof: A Kaehlerian manifold M is ***homogeneous*** if there is a connected Lie group which acts transitively and effectively on M as a group of diffeomorphisms and leaves the Kaehlerian form invariant. A Kaehlerian symmetric space is a homogeneous Kaehlerian manifold which is Remannian symmetric with respect to the Kaehlerian metric structure[3].

The Ricci curvature of a compact homogeneous Kaehlerian manifold M may not be positive semi-definite. For, let $A(S^{2n+1})$ be the automorphism group of S^{2n+1} with the almost Kaehlerian metric structure $\sum = (\emptyset, \xi . \eta, g)$. $A(S^{2n+1})$ is transitive. If $f = A(S^{2n+1})$ then

$$f^*[ag+(a^2-a)\eta \otimes \eta]=ag+(a^2-a)\, \eta \otimes \eta]$$

where ***a*** is a constant. Hence, $A(S^{2n+1})$ is also the automorphism group of the Kaehlerian structure $\sum = (\bar{\emptyset}, \bar{\xi} . \bar{\eta}, \bar{g})$. With

$$\bar{\emptyset} = \phi , \quad \bar{\xi} . = \frac{1}{a} \xi ,$$

$$\bar{\eta} = a\, \eta , \quad \bar{g} = a g + (a^{2} - a) \eta \otimes \eta .$$

A $\emptyset$ –basis for Σ can be modified to a ϕ-basis for $\overline{\sum}$ so from the sectional Curvatures $K_{\alpha\beta}$ and $\bar{K}_{\alpha\beta}$, of $\sum$ *and* $\overline{\sum}$,

$$\bar{K}_{ii^*} = \frac{1}{a} [K_{ii^*} + 3(1-a)] = \frac{4-3a}{a}.$$

If we put ***a*** = 2, then $\bar{K}_{ii^*} = -1$

Theorem (2.4): The fundamental group $\pi_1(M)$ of a compact symmetric Kaehlerian manifold M is finite.

Proof: Since a harmonic form on a compact symmetric space has vanishing covariant derivative, $B_1 = 0$ by Theorem (2.2). Let $M = G / K$ and assume that K is connected. Consider the exact homology sequence

$$0 \rightarrow \pi_1(K) \rightarrow \pi_1(G) \rightarrow \pi_1(M) \rightarrow 0.$$

Since $\pi_1(G)$ is abelian, so is $\pi_1(M)$. Hence

$$H_1(M, Z) \approx \pi_1(M) / [\pi_1(M), \pi_1(M)] \approx \pi_1(M).$$

Thus, since B1 = 0, $H_1(M, Z)$is a finite group since it is a finitely generated torsion group, so $\pi_1(M)$ is finite also.

If K is not connected, let K_0 be the connected component of the identity in K and consider the exact sequence

$$0 \rightarrow \pi_1(G / K_0) \rightarrow \pi_1(G / K) \rightarrow K/K_0 \rightarrow 0.$$

Since K is compact, K / K_0 is finite. Hence, since $\pi_1(G / K)$ is an extension of $\pi_1 (G / K_0)$ by K / K_0 it is finite[7].

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