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On $\mathbb{Q}$-fuzzy Prime Ideals Of $\mathbb{Q}$-rings

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Abstract

The notion of $\mathbb{Q}$-rings, $\mathbb{Q}$-ideals, $\mathbb{Q}$-prime ideals and their fuzzy counterparts are introduced and analyzed. We also give a characterization of $\mathbb{Q}$-fuzzy prime ideals of a fuzzy ring.

Key words: $\mathbb{Q}$-ring, $\mathbb{Q}$-fuzzy ring, $\mathbb{Q}$-fuzzy ideal, $\mathbb{Q}$-fuzzy prime ideal.

1. Introduction

Fuzzy Gamma rings were studied by Jun and Lee[3]. Dutta and Chanda[1] continued this investigation and obtained a characterization of fuzzy ideals of a Gamma ring. Later Shaoquan Sun[4] introduced the notions of fuzzy rings with operators, their ideals and quotient rings.

In this paper, as a continuation of these works, we introduce and analyze the notions of $\mathbb{Q}$-rings, $\mathbb{Q}$-ideals, $\mathbb{Q}$-prime ideals and their fuzzy counterparts. We also give a characterization of $\mathbb{Q}$-fuzzy prime ideals of a $\mathbb{Q}$-ring. Throughout this paper $\mathbb{Q}$[$\mathbb{R}$, $\mathbb{C}$] denote the set of all rational [real, complex] numbers and R denotes a general ring (R, +, $\circ$) unless otherwise stated. Algebraic terms and notations used in this work are as in Fraleigh[2].

2. $\mathbb{Q}$-Rings and $\mathbb{Q}$-Ideals :

2.1. Definition. Let R be a ring and $\mathbb{Q}$ be the set of all rational numbers. We say that R is a ***$\mathbb{Q}$-ring*** if for any $a \in R$ and $q \in \mathbb{Q}$, there exists a product qa, called the ***action of***

q on a such that for every a,b$\in$R and q$\in\mathbb{Q}$

(i) qa $\in$ R
(ii) q(a+b) = qa + qb, and
(iii) q(ab) = (qa)b = a(qb).

The above multiplication is referred to as the ***left action*** of $\mathbb{Q}$ on R. In a similar way, we can define the ***right action*** of $\mathbb{Q}$ on R. In this paper, we shall consider only left action. First we shall give some examples of $\mathbb{Q}$-rings.

2.2. Example. The rings $\mathbb{C}$ [$\mathbb{R}$, $\mathbb{Q}$] of all complex [real, rational] numbers; $\mathbb{C}^{n \times n}$ [$\mathbb{R}^{n \times n}$, $\mathbb{Q}^{n \times n}$] of all nxn square matrices of complex [real, rational] numbers; and $\mathbb{C}$[x], [$\mathbb{R}$[x], $\mathbb{Q}$[x]] of all polynomials in x with coefficients from $\mathbb{C}$[$\mathbb{R}$, $\mathbb{Q}$] are all Q-rings with respect to the usual multiplication.

Now we proceed to give some examples of rings which are not Q-rings. In all these examples, the action of Q on the rings is the usual multiplication.

2.3. Example. The ring of integers $\mathbb{Z}$; the ring of even integers E; and n$\mathbb{Z}$ are not $\mathbb{Q}$-rings.

2.4. Example. Let $\mathbb{Z}_n$ = {0, 1, 2, 3, … , n-1} and $\times_n$ and $+_n$ respectively denote multiplication modulo n and addition modulo n. Then ($\mathbb{Z}_n$, $+_n$, $\times_n$) is a ring; but it is not a $\mathbb{Q}$-ring.

2.5. Example. Let I[√2] = {a+b√2 : a, b are integers}. Then I[√2] is a commutative ring with identity with respect to usual addition and multiplication. But it is not a $\mathbb{Q}$-ring.

Recall that if R is a general ring and I $\subset$ R then I is said to be a ***left ideal*** of R if

(i) a-b $\in$ I and (ii) ra $\in$ I, $\forall$ a,b $\in$ I and r $\in$ R.

I is called ***right ideal*** of R if a-b$\in$I and ar$\in$I. We say that I is an ***ideal*** of R if it is both a left ideal and a right ideal. For any ring R, {0} and R itself are ideals. They are called the ***trivial ideals***. Any ideal different from these two are called ***non-trivial (or proper) ideal***. Now we turn to ideals in $\mathbb{Q}$-rings.

2.6. Definition. Let R be a $\mathbb{Q}$-ring and I be a [left, right] ideal of R. Then I is said to be a ***[left, right] $\mathbb{Q}$-ideal*** of R if qa$\in$I, $\forall$ q$\in\mathbb{Q}$ and a$\in$I.

In any $\mathbb{Q}$-ring R, the trivial ideals {0} and R itself are $\mathbb{Q}$-ideals. In $\mathbb{Q}$-rings $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{Q}$, there are only the trivial Q-ideals. But there are !-rings possessing non-trivial $\mathbb{Q}$-ideals. The following is an example.

2.7. Example. Let R = $\mathbb{C}$[x], the ring of polynomials in x with complex coefficients and let I = (x^2 + 1) be the ideal generated by the polynomial x^2 + 1. Then every element of I has the form f(x)(x^2+1), where f(x)$\in\mathbb{C}$[x]. Obviously, q f(x)$\in\mathbb{C}$[x], $\forall$ q$\in\mathbb{Q}$, and so q f(x) (x^2+1)$\in$I. Hence I is a $\mathbb{Q}$-ideal of R. Note that for A, B $\subset$ R, AB = {ab : a$\in$A, b$\in$B}

2.8. Definition. Let P be a proper $\mathbb{Q}$-ideal of a $\mathbb{Q}$-ring R. Then P is said to be a ***$\mathbb{Q}$-prime ideal*** if for every pair of $\mathbb{Q}$-ideals A and B of R

AB $\subset$ P $\Rightarrow$ either A $\subset$ P or B $\subset$ P or equivalently

$ab \in P \Rightarrow$ either $a \in P$ or $b \in P$.

2.9. Example. Consider the $\mathbb{Q}$-ring $\mathbb{R}[x]$ of all polynomials in x with real coefficients. Let P = (x+1), be the ideal generated by x+1 so that every element $r(x) \in P$ has the form r(x) = f(x) (x+1), where $f(x) \in \mathbb{R}[x]$.

Suppose $ab \in P$. Then ab = f(x) (x+1), for some $f(x) \in \mathbb{R}[x]$. This implies that either a or b is a polynomial multiple of (x+1). That is, either $a \in P$ or $b \in P$. Hence P is a $\mathbb{Q}$-prime ideal of $\mathbb{R}[x]$.

3. $\mathbb{Q}$-Fuzzy rings and ideals :

Let R be a ring and A be a ***fuzzy set*** on R (*i.e.* A is a function from R to the closed interval [0,1]). Then A is called a ***fuzzy ring*** on R if for every x, y $\in$ R :

$A(x-y) \geq \min\{A(x), A(y)\}$ and

$A(xy) \geq \min\{A(x), A(y)\}$.

For any $\alpha \in [0,1]$ the set

$\alpha_A = \{x \in R : A(x) \geq \alpha\}$

Is called the ***α-level set*** (or ***α-cut***) of A. It is well known that a non-constant fuzzy set A on R is a fuzzy ring if, and only if, α_A is a subring of R, $\forall\ \alpha \in Im(A)$.

A fuzzy ring A on R is called a ***fuzzy left ideal*** of R if

$A(xy) \geq A(y)$, $\forall$ x, y $\in$ R

and a ***fuzzy right ideal*** if

$A(xy) \geq A(x)$, $\forall$ x, y $\in$ R.

We say that A is a ***fuzzy ideal*** of R if it is both a fuzzy left ideal and a fuzzy right ideal. In other words, a fuzzy set A on R is a fuzzy ideal of R if $\forall$ x, y $\in$ R

(i) $A(x-y) \geq \min\{A(x), A(y)\}$ and

(ii) $A(xy) \geq \max\{A(x), A(y)\}$.

3.1. Definition. Let R be a $\mathbb{Q}$-ring and A be a fuzzy ring on R. Then A is said to be a ***$\mathbb{Q}$-fuzzy ring*** on R if

$A(qx) \geq A(x)$, $\forall\ x \in R$ and $q \in \mathbb{Q}$.

3.2. Example. Consider the $\mathbb{Q}$-ring $\mathbb{C}[x]$ of all polynomials in x having complex coefficients. Define $A : \mathbb{C}[x] \to [0, 1]$ by

A[f(x)] = 1, if x+1 divides f(x)

= 0.4, otherwise.

Then A is a fuzzy ring on $\mathbb{C}[x]$. Now for any $q \in \mathbb{Q}$ and $f(x) \in \mathbb{C}[x]$, x+1 divides q f(x) if, and only if, x+1 divides f(x). Hence A[q f(x)] has the same value as A[f(x)], Therefore, we can write

$A[q\ f(x)] \geq A[f(x)]$, $\forall\ q \in \mathbb{Q}$ and $f(x) \in \mathbb{C}[x]$.

Hence A is a $\mathbb{Q}$-fuzzy ring on $\mathbb{C}[x]$.

It may be recalled that if A and B are fuzzy sets on R, then $A \cup B$ and $A \cap B$ are defined by

$(A \cup B)(x) = \max\{A(x), B(x)\}$ and

$(A \cap B)(x) = \min\{A(x), B(x)\}$, $\forall\ x \in R$.

Further if A and B are fuzzy rings on R, then $A \cap B$ is a fuzzy ring on R, but $A \cup B$ need not be a fuzzy ring. This extends to $\mathbb{Q}$-fuzzy rings also. The case of $A \cup B$ is trivial, since it fails to be even a fuzzy ring. In the following proposition we prove the case of $A \cap B$.

3.3. Proposition. Let A and B be $\mathbb{Q}$-fuzzy rings on a $\mathbb{Q}$-ring R. Then $A \cap B$ is a $\mathbb{Q}$-fuzzy ring on R.

Proof: Since A and B are fuzzy rings on R, $A \cap B$ also is a fuzzy ring on R. Now let $q \in \mathbb{Q}$ and $x \in R$. Since A and B are $\mathbb{Q}$-fuzzy rings on R, we have

$A(qx) \geq A(x)$ and $B(qx) \geq B(x)$

$$\Rightarrow \min\{A(qx), B(qx)\} \geq \min\{A(x), B(x)\}$$

$$\Rightarrow (A \cap B)(qx) \geq (A \cap B)(x).$$

Hence $A \cap B$ is a $\mathbb{Q}$-fuzzy ring on R∎

If $\{A_i : i \in I\}$ is an arbitrary collection of fuzzy sets on R, then their intersection $\cap_{i \in I} A_i$ is defined by

$$(\cap_{i \in I} A_i)(x) = \inf_{i \in I} A_i(x), \ \forall\, x \in R.$$

The arguments in the above proof extends to arbitrary intersection also. Hence we get the following as a corollary to the last proposition.

Corollary. Any intersection of $\mathbb{Q}$-fuzzy rings is a $\mathbb{Q}$-fuzzy ring∎

3.4. Proposition. Let R be a $\mathbb{Q}$-ring and A be a fuzzy ring on R. Then A is a $\mathbb{Q}$-fuzzy ring on R if, and only if, A_t is a $\mathbb{Q}$-subring of R, $\forall\, t \in [0, 1]$ for which $A_t \neq \varphi$.

Proof : Suppose A is a $\mathbb{Q}$-fuzzy ring on R. Let $t \in [0, 1]$ such that $A_t \neq \varphi$. Since A is a fuzzy ring on R, A_t is a subring of R. Now let $q \in \mathbb{Q}$ and $x \in A_t$. Then $A(x) \geq t$. Since A is a $\mathbb{Q}$-fuzzy ring, $A(qx) \geq A(x)$. Hence $A(qx) \geq t$. Therefore, $qx \in A_t$. This proves that A_t is a subring of R.

Conversely, suppose A_t is a $\mathbb{Q}$-subring of R, $\forall\, t \in [0, 1]$ for which $A_t \neq \varphi$. Then A_t is a subring of R for every such t. Therefore A is a fuzzy subring of R. Further, $\forall\, q \in \mathbb{Q}$ and $x \in R$, since each A_t is a $\mathbb{Q}$-subring of R,

$$x \in A_t \Rightarrow qx \in A_t, \ \forall\, q \in \mathbb{Q}.$$

Hence,

$$A(x) \geq t \Rightarrow A(qx) \geq t, \ \forall\, t.$$

This is possible only if

$$A(qx) \geq A(x), \ \forall\, q \in \mathbb{Q} \text{ and } x \in R.$$

Therefore A is a fuzzy subring of R∎

3.5. Definition. Let R be a $\mathbb{Q}$-ring A be a $\mathbb{Q}$-fuzzy ring on R. Then A is called a ***$\mathbb{Q}$-fuzzy [left, right] ideal*** of R if A is also a fuzzy [left, right] ideal of R.

3.6. Example. Consider the $\mathbb{Q}$-fuzzy ring A on $\mathbb{C}[x]$ given in example 3.2. It is also a fuzzy ideal of $\mathbb{C}[x]$. Hence A is a $\mathbb{Q}$-fuzzy ideal of $\mathbb{C}[x]$.

We give below two propositions regarding $\mathbb{Q}$-fuzzy ideals, omitting their proofs which are similar to the proofs of proposition 3.3 and 3.4.

3.7. Proposition. Let R be a $\mathbb{Q}$-ring and A, B be $\mathbb{Q}$-fuzzy [left, right] ideals of R. Then $A \cap B$ is a $\mathbb{Q}$-fuzzy]left, right] ideal of R∎

Corollary. Any intersection of $\mathbb{Q}$-fuzzy [left, right] ideals is a $\mathbb{Q}$-fuzzy [left, right] ideal∎

3.8. Proposition. Let R be a $\mathbb{Q}$-ring and A be a fuzzy set on R. Then A is a $\mathbb{Q}$-fuzzy [left, right] ideal of R if, and only if, A_t is a [left, right] $\mathbb{Q}$-ideal of R, $\forall\, t \in [0, 1]$ for which $A_t \neq \varphi$∎

4. $\mathbb{Q}$-Fuzzy prime ideals :

4.1. Definition. If A, B are fuzzy sets on R, Then their ***product*** AB is defined as follows: $\forall x \in R$

$(AB)(x) = \sup_{x=yz} \{\min[A(y), B(z)]\}$
$= 0$, if x is not expressible as $x = yz$.

4.2. Definition. Let R be a $\mathbb{Q}$-ring and A be a non-constant $\mathbb{Q}$-fuzzy ideal of R. Then A is said to be a $\mathbb{Q}$-***fuzzy prime ideal*** if for any two $\mathbb{Q}$-fuzzy ideals B and C of R

$$BC \subset A \Rightarrow \text{ either } B \subset A \text{ or } C \subset A.$$

4.3. Example. The $\mathbb{Q}$-fuzzy ideal A of $\mathbb{C}[x]$ given in example 3.2 is $\mathbb{Q}$-fuzzy prime ideal.

4.4. Example. Consider the $\mathbb{Q}$-ring $\mathbb{R}[x]$ of all polynomials in x having real coefficients. Define the fuzzy set F on $\mathbb{R}[x]$ by

$F[f(x)] = 1$, if (x^2+1) divides $f(x)$
$= 0.2$, otherwise.

Then F is a $\mathbb{Q}$-fuzzy prime ideal of $\mathbb{R}[x]$.

We now give below a characterization of $\mathbb{Q}$-fuzzy prime ideals of a general $\mathbb{Q}$-ring R.

4.5. Theorem. Let I be a $\mathbb{Q}$-ideal of a $\mathbb{Q}$-ring R, $\alpha \in [0, 1)$ and F be a fuzzy set on R defined by

$F(x) = 1$, if $x \in I$
$= \alpha$, otherwise.

Then F is a $\mathbb{Q}$-fuzzy prime ideal of R if, and only if, I is a $\mathbb{Q}$-prime ideal of R.

Proof: Suppose I is a $\mathbb{Q}$-prime ideal of R. We have to prove that F is a $\mathbb{Q}$-fuzzy prime ideal of R. Obviously, F is non-constant. First we will prove that $F(a-b) \geq \min\{F(a), F(b)\}$, $\forall$ a, b $\in$ R.

Case (i): If $\min\{F(a), F(b)\} = \alpha$, then by the definition of F

$F(a-b) = 1$ or $\alpha \geq \min\{F(a), F(b)\}$.

Case (ii): If $\min\{F(a), F(b)\} = 1$, then $F(a) = F(b) = 1$. Therefore a, b $\in$ I. Since I is an ideal a-b $\in$ I. Hence $F(a-b) = 1 = \min\{F(a), F(b)\}$.

Thus in both cases, $F(a-b) \geq \min\{F(a), F(b)\}$.In a similar way we get

$F(ab) \geq F(a)$ and $F(ab) \geq F(b)$

and hence

$F(ab) \geq \max\{F(a), F(b)\}$.

Therefore F is a fuzzy ideal of R. Also since I is a $\mathbb{Q}$-ideal of R, F is a $\mathbb{Q}$-fuzzy ring on R, and hence is a $\mathbb{Q}$-fuzzy ideal of R.

Now, suppose that there exist two $\mathbb{Q}$-fuzzy ideals A and B of R such that $AB \subset F$, $A \not\subset F$ and $B \not\subset F$. Then there exist x, y $\in$ R such that $A(x) > F(x)$ and $B(y) > F(y)$. Since $AB \subset F$, this implies that $F(x) = F(y) = \alpha$. Hence x, y, xy $\notin$ I. Therefore by the definition of F, $F(x) = F(y) = F(xy) = \alpha$. Now,

$(AB)(xy) = \sup\{\min[A(x), B(y)]\}$
$\geq \min[A(x), B(y)]$
$> \min[F(x), F(y)]$, since $A(x) > F(x)$ and $B(y) > F(y)$
$= \alpha = F(xy)$

Thus there exist x, y $\in$ R such that $(AB)(xy) > F(xy)$. This contradicts the hypothesis that $AB \subset F$. Hence

$$AB \subset F \Rightarrow \text{ either } A \subset F \text{ or } B \subset F.$$

Therefore F is a $\mathbb{Q}$-fuzzy prime ideal of R.

Conversely, suppose F is a $\mathbb{Q}$-fuzzy prime ideal of R. We have to prove that I is a $\mathbb{Q}$-prime ideal of R. If not, we can find two $\mathbb{Q}$-ideals A and B of R such that $AB \subset I$, but $A \not\subset I$ and $B \not\subset I$. Hence there exist elements p, q $\in$ R such that $p \in A$, but $p \notin I$ and $q \in B$, but $q \notin I$. Define fuzzy sets F_1 and F_{-2} on R by

$F_1(x) = 1$, if $x \in A$ and $F_1(x) = \alpha$, if $x \notin A$
$F_2(x) = 1$, if $x \in B$ and $F_2(x) = \alpha$, if $x \notin B$.

Then F_1 and F_2 are $\mathbb{Q}$-fuzzy ideals of R such that $F_1F_2 \subset F$. Also, $F_1(p) = 1 > \alpha = F_2(p)$. Therefore, $F_1 \not\subset F$. Similarly we get $F_2 \not\subset F$. This is a contradiction, since F is a $\mathbb{Q}$-fuzzy prime ideal of R. Hence I is a $\mathbb{Q}$-prime ideal of R∎

Corollary. The characteristic function c_I of a $\mathbb{Q}$-ideal I of a $\mathbb{Q}$-ring R is a $\mathbb{Q}$-fuzzy prime ideal if, and only if, I is a $\mathbb{Q}$-prime ideal of R.

Proof: This follows from last theorem by taking $\alpha = 0$∎

4.6. Remark. $\mathbb{Q}$-fuzzy prime ideals can be formed only as in the above theorem. Hence if μ is a $\mathbb{Q}$-fuzzy prime ideal of R, then $Im(\mu) = \{1, \alpha\}$, for some $\alpha \in [0, 1)$.

5. References

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