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Finite Sets, Separation Axioms, and Equivalences of Open and Closed Images

CHARLES DORSETT

Texas A&M University-Commerce

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Abstract

Within this paper nonempty finite sets are further characterized using closed images and the R_0 and R_1 separation axioms, and equivalences of open and closed images are given for T_1 and R_0 spaces.

Key words: finiteness, open and closed images, separation axioms, and discreteness.

Subject classification: 54A05, 54C10, 54D10, 54D15.

Introduction

1. Motivation and Direction: During the Summer 2007, within an introductory graduate level topology class, the students were asked to prove or disprove the open image of a T_0 space is T_0. The students' intuition was good; they believed the statement to be false, but in trying to create a counterexample, each time the students unsuccessfully started with a finite T_0 space. Thus the question: "For finite, T_0 spaces, must open images of the space be T_0?" was raised. The investigation of this question has led to many topological characterizations of nonempty sets using open images and separation axioms including the following.

Theorem 1.1. Let X be a nonempty set. Then the following are equivalent: (a) X is finite, (b) for each topology T on X for which (X,T) is T_0, each open image of (X,T) is T_0, (c) for each topology T on X for which (X,T) is T_1, each open image of (X,T) of (X,T) is T_1, and (d) for each topology T on X for which (X,T) is T_2, each open image of (X,T) is T_2[2].

In 1943 T_1 spaces were generalized to R_0 spaces[5] and in 1961 T_2 spaces were generalized to R_1 spaces[1], which raised the question: "Can nonempty finite sets be characterized using open images and the R_0 or R_1 separation axioms?" leading to additional discoveries.

Definition 1.1. A space (X,T) is R_0 iff for each closed set C and each x not in C,

$C \cap Cl(\{x\}) = \phi$[5].

Definition 1.2. A space (X,T) is R_1 iff for x, y in X such that $Cl(\{x\}) \neq Cl(\{y\})$, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$[1].

Theorem 1.2. Let (X,T) be a space. Then every open image of (X,T) is R_0 iff every open image of (X,T) is T_1[3].

Theorem 1.3. Let X be a nonempty set. Then the following are equivalent: (a) X is finite, (b) for each topology T on X for which (X,T) is R_0, each open image of (X,T) is R_0, and (c) for each topology T on X for which (X,T) is R_1, each open image of (X,T) is R_1[3].

In late February, 2012, while continuing the investigation of nonempty finite sets, open images, and separation axioms, the question surfaced of what would happen if open images were replaced by closed images? The investigation of that question includes the results below, which include a new characterization of T_1 spaces.

Theorem 1.4. Let X be a nonempty set. Then the following are equivalent: (a) X is finite, (b) for each topology T on X for which (X,T) is T_0, each closed image of (X,T) is T_0, (c) for each T_1 topology on X, for each closed image (Y,S) of (X,T), (Y,S) is T_1 with S = P(Y), the power set of Y, and (d) for each topology T on X for which (X,T) is T_1, each function f from (X,T) onto a space (Y,S) is closed iff it is open[4].

Within the paper cited above[4], T_1 in the statement of Theorem 1.4 parts (c) and (d) was replaced by each of T_2, T_3, $T_{31/2}$, T_4, completely normal T_1, and metrizable.

Theorem 1.5. A space (X,T) is T_1 iff each closed image of (X,T) is T_1[4].

Below the questions of what happens when T_1 in the two results above is replaced by R_0 and R_1 are investigated. Because of the questions, in the characterizations above for a nonempty finite set X, requirements were made on all topologies on X satisfying certain properties, which raised the question: (1) "For a nonempty set X, what would happen if the requirements are on only one topology on X satisfying the properties?" This question is addressed below.

2. Closed Images and R_0, R_1, and Other Spaces.

Theorem 2.1. Let (X,T) be a space. Then (a) every closed image of (X,T) is R_0 iff (b) every closed image of (X,T) is T_1.

Proof: Since T_1 implies R_0, (b) implies (a).

(a) implies (b): Suppose there is a closed function f from (X,T) onto a space (Y,S) for which (Y,S) is not T_1. Let u be in Y for which $Cl(\{u\}) \neq \{u\}$. Let $v \in Cl(\{u\}) \setminus \{u\}$. Then $B = S \cup \{\{u\}\}$ is a base for a topology W on Y and f is a closed function from (X,T) onto (Y,W), which implies (Y,W) is R_0, but $C = Y \setminus \{u\}$ is closed in (Y,W), $u \notin C$, and $v \in Cl_W(\{u\}) \cap C$, which contradicts (Y,W) is R_0.

Thus each closed image of (X,T) is T_1.

Theorem 2.1 can be combined with Theorem 1.5 above to give an additional characterization of T_1 spaces.

Corollary 2.1. Let (X,T) be a space. Then (X,T) is T_1 iff every closed image of (X,T) is R_0.

Theorem 2.2. Let (X,T) be a space. Then (a) every closed image of (X,T) is R_1 iff (b) every closed image of (X,T) is T_2.

Proof: Since T_2 implies R_1[1], (b) implies (a).

(a) implies (b): Since R_1 implies R_0[1], then, by Theorem 2.1, every closed image of (X,T) is R_1 and T_1, which implies (X,T) is T_2[1].

Theorem 2.3. Let X be a nonempty set. Then the following are equivalent: (a) X is finite, (b) for each topology T on X for which each closed image (Y,S) of (X,T) is R_0, S = P(Y), and (c) for each topology T on X for which each closed image of (X,T) is R_0, each function f from (X,T) onto a space (Y,S) is closed iff it is open.

Proof: (a) implies (b): Let T be a topology on X for which each closed image (Y,S) of (X,T) is R_0. Then each closed image (Y,S) of (X,T) is T_1, which implies (X,T) is T_1, and, since X is finite, by Theorem 1.4, S = P(Y).

(b) implies (c): Let T be a topology on X for which each closed image of (X,T) is R_0. Let (Y,S) be a closed image of (X,T) and let f be a function from (X,T) onto (Y,S). Then S = P(Y) and f is both closed and open.

(c) implies (a): Let T be a topology on X for which each closed image of (X,T) is T_1. Then each closed image of (X,T) is R_0 and each function f from (X,T) onto a space (Y,S) is closed iff it is open. Thus, by Theorem 1.4, X is finite.

Theorem 2.4. Let X be a nonempty set and let P be anyone of the separation axioms R_1, regular, completely regular, normal R_0, completely normal R_0, and pseudometrizable. Then the following are equivalent: (a) X is finite, (b) for each topology T on X for which each closed image (Y,S) of (X,T) has separation axiom P, S =P(Y), and (c) for each topology T on X for which each closed image of (X,T) has separation axiom P, each function f from (X,T) onto a space (Y,S) is closed iff it is open.

The proof is straightforward using the information given above and a proof similar to that of Theorem 2.3, and is omitted.

3. The Requirements Restricted to Only One Topology.

Theorem 3.1. Let (X,T) be a space. Then the following are equivalent: (a) T = P(X), (b) for each image Y of X and function f from X onto Y, there is only one topology S on Y for which f is open, (c) for each image Y of X and function f from X onto Y, the only topology S on Y for which f is open is S = P(Y), and (d) each open image of (X,T) is T_1.

Proof: (a) implies (b): Let Y be an

image of X. Let f be a function from X onto Y. Let S be a topology on Y such that f is open from (X,T) onto (Y,S). Since singleton sets are open in (X,T), then singleton sets are open in (Y, S), which implies S = P(Y). Since f is an open function from (X,T) onto (Y,P(Y)), then there is only one topology on Y for which f is open.

(b) implies (c): Let Y be an image of X and let f be a function from X onto Y. Let S be a topology on Y for which f is an open function from (X,T) onto (Y,S). Since f is an open function from (X,T) onto (Y,P(Y)) and there is only one topology W on Y for which f is an open function from (X,Y) onto (Y,W), then S = P(Y).

(c) implies (d): Let (Y,S) be an open image of (X,T). Then S=P(Y) and (Y,S) is T_1.

(d) implies (a): Suppose T $\neq$ P(X). Let x $\in$ X such that {x} $\notin$ T. For each open set O containing x, let $x_O \in$ O such that x $\neq$ x_O and let Z = {x_O: x $\in$ O $\in$ T}. Let z $\notin$ X and let Y = {z} $\cup$ (X \ Z). Let S be the topology on Y with base {U $\in$ T: U $\cap$ Z= ϕ} $\cup$ {{z}} $\cup$ {(U \ Z) $\cup$ {z}: U $\in$ T and U $\cap$ Z $\neq$ ϕ}. Let f be the function from (X,T) onto (Y,S) defined by f(u) = u if u $\notin$ Z and f(u) = z for all u $\in$ Z. Then f is open and onto, but every S-open set containing x contains z and (Y,S) is not T_1, which is a contradiction. Hence T = P(X).

Theorem 3.2. Let (X,T) be a space. Then the following are equivalent: (a) T = P(X), (b) each open image of (X,T) is R_0, and (c) (X,T) is T_1 and a function f from (X,T) onto a space (Y,S) is open iff it is closed.

Proof: Clearly from the information and arguments above (a) and (b) are equivalent and (a) implies (c).

(c) implies (a): Let (Y,S) be an open image of the T_1 space (X,T). Let f be an open function from (X,T) onto (Y,S). Then f is a closed function from (X,T) onto (Y,S), which implies (Y,S) is T_1. Thus every open image of (X,T) is T_1 and by Theorem 3.1., T = P(X).

Theorem 3.3. Let (X,T) be a space. Then the following are equivalent: (a) T is the indiscrete topology on X and (b) for each image Y of X and each function f from X onto Y, for each topology S on Y, f, a function from (X,T) onto (Y,S), is simultaneously open and closed.

Proof: (a) implies (b): Let Y be an image of X and let f be a function from X onto Y. Let S be a topology on Y. Since the only open sets in (X,T) are the empty set and X, then f, a function from (X,T) onto (Y,S) is open. Similarly f is closed.

(b) implies (a): Suppose T is not the indiscrete topology on X. Let S be the indiscrete topology on X. Then the identity function from (X,T) onto (X,S) is neither open nor closed, which is a contradiction.

Theorem 3.4. Let (X,T) be a space. Then the following are equivalent: (a) T = P(X) or T is the indiscrete topology on X and (b) (X,T) is R_0 and a function f from (X,T) onto a space (Y,S) is open iff it is closed.

Proof: (a) implies (b):If T = P(X), then by Theorem 3.2, (X,T) is T_1 and thus R_0 and a function f from (X,T) onto a space (Y,S) is open iff it is closed. If T is the indiscrete topology on X, then (X,T) is R_0 and, by Theorem 3.3, the remainder of part (b) is satisfied.

(b) implies (a): If (X,T) is T_1, then by Theorem 3.2, T = P(X). Thus consider the case that (X,T) is R_0 and not T_1. Let $x \in X$ such that Cl({x}) is not {x}. Then X = Cl({x}), for suppose not. Let $u \in$ Cl({x}}, $u \neq x$. Let $v \in X \setminus Cl(\{x\})$. Since (X,T) is R_0, $Cl(\{x\}) \cap Cl(\{v\}) = \phi$. Let Y = X \ Cl({v}) and let S be the topology on Y with base $\{O \in T: O \subseteq X \setminus Cl(\{v\})\} \cup \{\{x\}\}$ and let f be the function from (X,T) onto (Y,S) defined by f(w) = w for all $w \in Y$ and f(w) = x for all $w \in Cl(\{v\})$. Let $O \in T$. If $O \subseteq Y$, then $f(O) = O \in S$. Thus consider the case that $O \cap Cl(\{v\}) \neq \phi$. Then $f(O) = (O \setminus Cl(\{v\}) \cup \{x\} \in S$. Thus f is open, but not closed; f(Cl({v})) = {x} and every S-open set containing u contains x. Hence Cl({x}) = X. Let C be a nonempty closed set in (X,T). Let $z \in C \cap Cl(\{x\})$. Since (X,T) is R_0, Cl({z}) = Cl({x}) = X and since $Cl(\{z\}) \subseteq C$, C = X. Thus T is the indiscrete topology on X.

As in the case of Theorem 2.4 above, the information and proofs methods above can be used to give the next result in this paper, which is given without proof.

Theorem 3.5. Let X be a nonempty set and let P be anyone of the separation axioms R_1, regular, completely regular, normal R_0, completely normal R_0, and pseudometrizable. Then the following are equivalent: (a) T = P(X) or T is the indiscrete topology on X and (b) (X,T) has separation axiom P and a function f from (X,T) onto a space (Y,S) is open iff it is closed.

There are non-R_0 spaces for which a function f from (X,T) onto a space (Y,S) is open iff it is closed. A place to look for such an example would be topological spaces with two distinct elements.

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