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A Closedown time dependent Bulk queue with multiple vacations

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Abstract

In this paper a Closedown time concent is introduced in a Bulk queuing model closedown time dependent a $M^X/G(a,b)$/queuing system with multiple vacation is considered. After completing a service, if the queue length is ξ, where ξ <a, then the server performs closedown work. After that the server leaves for multiple vacation of random length, irrespective of queue length. After a vacation, when it returns. If the queue length is less than ‘a’, it leaves for another vacation and so on, until he finds ‘a’ customers in the queue. After a vacation, if the server finds at least ‘a’ customers waiting for service, say , then he serves a batch of size min (ξ,b) customers, where b $\geq$a. Various Characteristics of the queueing system and a cost model with the numerical result for a particular case of the model are presented in this paper.

Key words : Bulk queue, multiple vacations, closedown times.

1.0 Introduction

Server vacation models are useful for the systems in which a server wants to utilize the idle time for different purposes. Application of server vacation models can be found in manufacturing systems, designing of local area networks and data communication systems etc.

Queueing problems with server vacations have been analysed by various authors with several combinations. Survey on queueing systems with server vacations can be found in Doshi, B.T.[2]. Lee, H.S.[6] has developed a procedure to calculate the system size probabilities for a bulk queueing model. Lee *et al*.[8] have analysed a batch arrival queue with N-policy, but considered single service and single vacation only. A $M^X/G/1$ queue with N-policy and multiple vacations is analysed by Lee *et al*.[7], in which arrivals occur in bulk and servi e is done one at a time, Chae, K.C. and Lee, H.W.[3], have analysed a $M^X/G/1$ vacation model with N-policy and discussed heuristic interpretation of mean waiting time. Krishna Reddy *et al*.[5] have analysed a bulk queueing model and multiple vacations with setup time. They derived the expected number of customers in the queue at an arbitrary time epoch and obtained other measures. Arumuganathan, R. analyses[1] some bulk queueing systems with multiple vacations and presented various performance measures. He has given an analytic approach to express the arbitrary constants in a compact form. For further study on vacation models one can refer the book by Takagi[10].

However, there has been a very few works on queueing systems with closedown time. A M/G/1 queue is analysed by Takagi[10], considering closedown time and setup time. He obtained performance measures too. It is observed that most of the studies on vacation queues are concentrated only on single server or single arrival and single vacation. Once the arrival occurs in bulk one can expect that the service can also be done in bulk.

This paper concentrations such a queueing system with closedown time. Dependent bulk queue with multiple vacations. In practice one can expect that the server require some amount of closedown time after completing the service. That is, after completing a service, if the queue length is $\xi < a$, then the server performs closedown work. After that, the server leaves for multiple vacations of random length, irrespective of queue length. When he returns, if the queue length is less than 'a', he leaves for another vacation and so on, until he finds 'a' customers in the queue. However, if the server finds at least 'a' customers waiting for service, say ξ (ξ>a), then he serves a batch of min (ξ,b) customers, where $b \geq a$.

It may be remarked here that our paper addresses the following points. Closedown time concept is introduced in a bulk queueing model. We obtain probability generating function of queue length distributions at an arbitrary time epoch. We have developed a cost model for the proposed queueing system. Important contribution is the study of cost model for a practical situation and how the results are useful regarding the decision making to optimize the cost.

The paper is organized as follows: in Section 2.0, we discuss the queueing problem with practical example and develop the system equations. In Section 3.0, we obtain probability generating function of the queue length distribution in a steady state condition and various performance measures of the queueing system is also be presented. Cost model is obtained analytically in Section. 4.0 A computational study is presented in Section 5.0. Conclusions and scope for further research are given in Section 6.0.

2.0 Description of the Model and General Equations :

We consider a situation in a Globe Valve manufacturing industry where the single server model is applied. In Globe Valve manufacturing industry, after turning operation the components arrive from job shop in batches to CNC turning center for facing and turning processes. The operator of CNC turning center starts the process only if the minimum batch quantity available. After processing a batch, if the number of components is not sufficient to process, then the operator leaves for arranging the toolings, writing the coding etc. to utilize his idle time. Before leaving, the operator must perform the works like closing the door, checking the toolings and etc. When the operator returns to CNC turning center, if the number of available components is less than a batch quantity, he remains in other works continuously until he finds enough quantity.

The above process can be modeled as follows. After completing a service, if the queue length is ξ, where $\xi < a$, then the server performs closedown work. Then the server leaves for multiple vacation of random length, irrespective of queue length. After a vacation. When he returns, if the queue length is less than 'a', he leaves for another vacation and so on, until he finds 'a' customers in the queue. However, on his return if the server finds at least 'a' customers waiting for service, say ξ, then he serves a batch of min (ξ, b) customers, where $b \geq a$.

Let X be the group size random variable of the arrival, g_k be the probability that 'k' customers arrive in a batch and X (z) be its probability generating function. Let S (.), V (.) and C (.) be the cumulative distribution of the service time, vacation time and closedown time respectively. s (x), v (x) and c (x) be the probability density function of service time, vacation time and closedown time respectively. At an arbitrary time, $S^0(t)$ denotes the remaining service time of a server in a batch, $V^0(t)$, $C^0(t)$ denotes the remaining vacation time, closedown time of a server respectively. Let us denote the Laplace-Stiltje's transforms of S, V and C and respectively.

Using Supplementary variables one can convert no-Markovian models into Markovian models. The Supplementary variables technique introduced by Cox, D.R.[4] was followed by Lee, H.S.[6]. He introduced an effective techniques for solving queueing models using supplementary variables. We use the technique that procedure for solving our model.

We define

$Y(t) = (0)[1]\{2\}$ if the server is on (busy) [closedown job] {vacation}

Z (t) = j if the sever is on j^{th} vacation

$N_s(t)$ = Number of customers in the server

$N_q(t)$ = Number of customers in the queue.

Let

$$P_{ij}(x,t)dt = P\{Ns(t) = i, N_q(t) = j, x \leq S^0(t) \leq x + dt, Y(t) = 0\}, a \leq x \leq b, j \leq 0,$$

which means that there are 'i' customers under service, 'j' customers in the queue, the server is busy with remaining service time x. In a similar manner we define.

$$Q_{jn}(x,t)dt = P\{Pq(t) = j, x \leq V^0(t) \leq x + dt, Y(t) = 2, z(t) = j\}, \ n \geq 0, j \geq 1.$$

$$C_n(x,t)\,dt = P\{Nq(t) = n, x \leq C^0(t) \leq x + dt, Y(t) = 1\}, n \geq 0.$$

We develop the system size equations. These equations provide the basis for the analysis given in sequel. These equations are obtained at time t + Δt considering all possibilities. One can not that when time t increased by Δt, the remaining service time, vacation time or closedown time will be reduced by x- Δt.

$$P_{i0}(x-\Delta t, t+\Delta t) = P_{i0}(x,t)(1-\lambda\Delta t) + \sum_{m=a}^{b} P_{mi}(0,t)\, s(x)\, \Delta t + \sum_{l=1}^{\infty} Q_{l0}(0,t) s(x) \Delta t \qquad a \le i \le b \tag{2.0.1}$$

$$P_{ij}(x-\Delta t, t+\Delta t) = P_{ij}(x,t)(1-\lambda\Delta t) + \sum_{k=1}^{j} P_{ij-k}(x,t)\lambda_{gk}\Delta t \quad j \ge 1, a \le i < b \tag{2.0.2}$$

$$P_{bj}(x-\Delta t, t+\Delta t) = P_{bj}(x,t)(1-\lambda\Delta t) + \sum_{k=1}^{j} P_{bj-k}(x,t)\lambda_{gk}\Delta t$$

$$+ \sum_{m=a}^{b} P_{mb+j}(0,t)s(x)\Delta t + \sum_{l=1}^{\infty} Q_{lb+j}(0,t)s(x)\Delta t \qquad j \ge 1 \tag{2.0.3}$$

$$C_n(x-\Delta t, t+\Delta t) = C_n(x,t)(1-\lambda\Delta t) + \sum_{m=a}^{b} P_{mn}(0,t)c(x)\Delta t$$

$$+ \sum_{k=1}^{n} C_{n-k}(x,t)\lambda_{gk}\Delta t \qquad n \le a-1 \tag{2.0.4}$$

$$C_n(x-\Delta t, t+\Delta t) = C_n(x,t)(1-\lambda\Delta t) + \sum_{k=1}^{n-a} C_{n-k}(x,t)\lambda_{gk}\Delta t, \qquad n \ge a \tag{2.0.5}$$

$$Q_{10}(x-\Delta t, t+\Delta t) = Q_{10}(x,t)(1-\lambda\Delta t) + C_0(0,t)v(x)\Delta t \tag{2.0.6}$$

$$Q_{1n}(x-\Delta t, t+\Delta t) = Q_{1n}(x,t)(1-\lambda\Delta t) + C_n(0,t)v(x)\Delta t$$

$$+ \sum_{k=1}^{n} Q_{1n-k}(x,t)\,\lambda_{gk}\,\Delta t, \quad n \ge 1 \tag{2.0.7}$$

$$Q_{j0}(x-\Delta t, t+\Delta t) = Q_{j0}(x,t)(1-\lambda\Delta t) + Q_{j-10}(0,t)v(x)\Delta t \qquad j \ge 0 \tag{2.0.8}$$

$$Q_{jn}(x-\Delta t, t+\Delta t) = Q_{jn}(x,t)(1-\lambda\Delta t) + Q_{j-1n}(0,t)v(x)\Delta t$$

$$+ \sum_{k=1}^{n} Q_{jn-k}(x,t)\,\lambda_{gk}\,\Delta t, \quad n < a-1,\ j \ge 2. \tag{2.0.9}$$

$$Q_{jn}(x-\Delta t, t+\Delta t) = Q_{jn}(x,t)(1-\lambda\Delta t) + \sum_{k=1}^{n} Q_{jn-k}(x,t)\,\lambda_{gk}\,\Delta, \quad n \ge a, j \ge 2. \tag{2.0.10}$$

3.0 Queue size Distribution

Dividing the equations (2.0.1-2.0.10) by Δt and letting the limit $\Delta t \to 0$, the steady state queue size equations are obtained as

$$P'_{i,0}(x) = -\lambda P_{i0}(x) + \sum_{m=a}^{b} P_{mi}(0)s(x) + \sum_{l=1}^{\infty} Q_{l0}(0)s(x) \quad a \le i \le b \tag{3.0.1}$$

$$P'_{i,j}(x) = -\lambda P_{ij}(x) + \sum_{k=1}^{j} P_{mj-k}(x)\lambda_{gk}, \quad a \le i \le b-1,\ j \ge 1 \tag{3.0.2}$$

$$P'_{bj}(x) = -\lambda P_{bj}(x) + \sum_{m=a}^{b} P_{mb+j}(0)s(x)$$

$$+ \sum_{k=1}^{j} P_{bj-k}(x)\lambda_{gk} + \sum_{l=1}^{\infty} Q_{lb+j}(0)s(x), \quad j \ge 1 \tag{3.0.3}$$

$$C'_{n}(x) = -\lambda C_{n}(x) + \sum_{m=a}^{b} P_{mn}(0)c(x) + \sum_{k=1}^{n} C_{n-k}(x)\lambda_{gk} \quad n \le a-1 \tag{3.0.4}$$

$$C'_{n}(x) = -\lambda C_{n}(x) + \sum_{k=1}^{n-a} C_{n-k}(x)\lambda_{gk} \quad n \ge a \tag{3.0.5}$$

$$Q'_{10}(x) = -\lambda Q_{10}(x) + C_0(0)v(x) \tag{3.0.6}$$

$$Q'_{1n}(x) = -\lambda Q_{1n}(x) + \sum_{k=1}^{n} Q_{1n-k}(x)\lambda_{gk} + C_n(0)v(x) \quad n \ge 0 \tag{3.0.7}$$

$$Q'_{j0}(x) = -\lambda Q_{j0}(x) + Q_{j-1,0}(0)v(x) \quad j \ge 2 \tag{3.0.8}$$

$$Q'_{jn}(x) = -\lambda Q_{jn}(x) + Q_{j-1,0}(0)v(x) + \sum_{k=1}^{n} Q_{jn-k}(x)\lambda_{gk} \quad n < a-1 \tag{3.0.9}$$

$$Q'_{jn}(x) = -\lambda Q_{jn}(x) + \sum_{k=1}^{n} Q_{jn-k}(x)\lambda_{gk} \quad j \ge 2,\ n \ge a \tag{3.0.10}$$

The Laplace Stiltje's transforms of $P_{ij}(x)$, $Q_{jn}(x)$, and $C_n(x)$ are defined as follows

$$\overline{P}_{in}(\theta) = \int_0^\infty e^{-\theta x} P_{ij}(x)\,dx, \quad \overline{Q}_{jn}(\theta) = \int_0^\infty e^{-\theta x} Q_{jn}(x)\,dx, \quad \overline{C}_n(\theta) = \int_0^\infty e^{-\theta x} C_n(x)\,dx$$

Taking Laplace Stiltje's transform on both sides of the equation (3.0.1-3.0.2), we get

$$\theta\overline{P}_{i0}(\theta)-P_{i0}(0)=\lambda\overline{P}_{i0}(\theta)-\sum_{m=a}^{b}P_{mi}(0)\overline{S}(\theta)-\sum_{l=1}^{\infty}Q_{lj}(0)\overline{S}(\theta)\quad a\le i\le b \tag{3.0.11}$$

$$\theta\overline{P}_{ij}(\theta)-P_{ji}(0)=\lambda\overline{P}_{ij}(\theta)-\sum_{k=1}^{j}P_{i\,j-k}(\theta)\lambda_{gk}\quad a\le i\le b-1,\ j\ge 1 \tag{3.0.12}$$

$$\theta\overline{P}_{bj}(\theta)-P_{bj}(0)=\lambda\overline{P}_{bj}(\theta)-\sum_{m=a}^{b}P_{mb+j}(0)\overline{S}(\theta)-\sum_{k=1}^{j}\overline{P}_{bj-k}(\theta)\lambda_{gk}$$
$$-\sum_{l=1}^{\infty}Q_{lb+j}(0)\overline{S}(\theta)\qquad j\ge 1 \tag{3.0.13}$$

$$\theta\overline{C}_n(\theta)-C_n(0)=\lambda\overline{C}_n(\theta)-\sum_{m=a}^{b}P_{mn}(0)\overline{S}(\theta)-\sum_{k=1}^{j}\overline{C}_{n-k}(\theta)\lambda_{gk}\qquad n\le a-1 \tag{3.0.14}$$

$$\theta\overline{C}_n(\theta)-C_n(0)=\lambda\overline{C}_n(\theta)-\sum_{k=1}^{n}\overline{C}_{n-k}(\theta)\lambda_{gk}\qquad n\ge a \tag{3.0.15}$$

$$\theta\overline{Q}_{10}(\theta)-\overline{Q}_{10}(0)=\lambda\overline{Q}_{10}(\theta)-C_0(0)\overline{V}(\theta) \tag{3.0.16}$$

$$\theta\overline{Q}_{1n}(\theta)-\overline{Q}_{1n}(0)=\lambda\overline{Q}_{10}(\theta)-\sum_{k=1}^{n}\overline{Q}_{1n-k}(\theta)\lambda_{gk}-C_n(0)\overline{V}(\theta)\qquad n>0 \tag{3.0.17}$$

$$\theta\overline{Q}_{j0}(\theta)-\overline{Q}_{j0}(0)=\lambda\overline{Q}_{j0}(\theta)-\overline{Q}_{j-10}(0)\overline{V}(\theta)\qquad j\ge 2 \tag{3.0.18}$$

$$\theta\overline{Q}_{jn}(\theta)-\overline{Q}_{jn}(0)=\lambda\overline{Q}_{jn}(\theta)-Q_{j-1n}(0)\overline{V}(\theta)-\sum_{k=1}^{n}\overline{Q}_{jn-k}(\theta)\lambda_{gk},$$
$$n\le a-1,\quad j\ge 2 \tag{3.0.19}$$

$$\theta\overline{Q}_{jn}(\theta)-\overline{Q}_{jn}(0)=\lambda\overline{Q}_{jn}(\theta)\sum_{k=1}^{n}\overline{Q}_{jn-k}(\theta)\lambda_{gk}\qquad n\ge a,\quad j\ge 2 \tag{3.0.20}$$

3.1 PGF of queue length distribution :

To apply the technique of Lee, H.S.[6], we define the probability generating functions as follows,

$$\overline{P}_i(z,\theta)=\sum_{j=0}^{\infty}\overline{P}_{ij}(\theta)z^j,\qquad P_i(z,0)=\sum_{j=0}^{\infty}P_{ij}(0)z^j,\qquad a\le i\le b$$

$$\overline{Q}_j(z,\theta)=\sum_{l=1}^{\infty}\overline{Q}_{lj}(\theta)z^j,\qquad Q_j(z,0)=\sum_{l=1}^{\infty}Q_{lj}(0)z^j,\qquad j\ge 1$$

$$\overline{C}(z,\theta)=\sum_{n=0}^{\infty}\overline{C}_n(0)z^n,\qquad C(z,0)=\sum_{n=0}^{\infty}C_n(0)z^n, \tag{3.1.1}$$

Multiplying (3.0.16) by z^0 and (3.0.17) by $z^n\,(n \geq 1)$, summing up from n = 0 to ∞ and using (3.1.1), we get

$$(\theta-\lambda X(z))\overline{Q}_1\,(z,\theta)=Q_1\,(z,0)-C(z,0)\,\overline{V}\,(\theta) \quad (3.1.2)$$

Multiplying (3.0.18) by z^0 (3.0.19) by $z^n\,(1 \leq n \leq a-1)$ and (3.0.20) by $z^n\,(n \geq a)$, summing up from n = 0 to ∞ and using (3.1.1), we get

$$(\theta-\lambda-\lambda X(z))\,\overline{Q}_j\,(z,\theta)=Q_j\,(z,0)$$
$$-\overline{V}(\theta)\sum_{n=0}^{a-1}Q_{j-1n}(0)\,z^n,\quad j\geq 2. \quad (3.1.3)$$

Multiplying (3.0.14) by $z^n\,(1\leq n\leq a-1)$, (3.0.15) by $z^n\,(n \geq a)$, summing up from n = 0 to ∞ and using (3.1.1), we get

$$(\theta-\lambda+\lambda X(z))\,\overline{C}\,(z,\theta),=C\,(z,0)$$
$$-\overline{C}(\theta)\sum_{n=0}^{a-1}\left(\sum_{m=a}^{b}P_{mn}(0)\overline{C}(\theta)\right)z^n. \quad (3.1.4)$$

Multiplying (3.0.11) by z^0, (3.0.12) by z^j ($j\geq 1$), taking the summation from j = 0 to ∞ and using (3.1.1), we get

$$(\theta-\lambda-\lambda X(z))\,\overline{P}_i\,(z,\theta)=P_i\,(z,0)-\overline{S}\,(\theta)$$
$$\sum_{m=a}^{b}P_{mi}(0)-\overline{S}(\theta)\sum_{l=1}^{\infty}Q_{li}\quad a\leq i\leq b-1 \quad (3.1.5)$$

Multiplying (3.0.11) by z^0, (3.0.13) by z^j ($j\geq 1$), taking the summation from j = 0 to ∞ and using (3.1.1), we get

$$(\theta-\lambda+\lambda X(z))\,\overline{P}_b\,(z,\theta)=P_b\,(z,0)$$
$$-\frac{\overline{S}(\theta)}{z^b}\left[\sum_{m=a}^{b}\left(P_m(z,0)-\sum_{j=0}^{b-1}P_{mj}(0)\,z^j\right)\right.$$
$$\left.+\sum_{l=1}^{\infty}\left(Ql(z,0)-\sum_{j=0}^{b-1}Q_{lj}(0)z^j\right)\right] \quad (3.1.6)$$

By substituting $\theta=\lambda-\lambda X(z)$ in the equation (3.1.2) – (3.1.6), we get

$$Q_1(z,0)=\overline{V}(\lambda-\lambda X(z))C(z,0) \quad (3.1.7)$$

$$Q_j(z,0)=\overline{V}(\lambda-\lambda X(z))\sum_{n=0}^{a-1}Q_{j-1n}(0)\,z^n\quad j\geq 2 \quad (3.1.8)$$

$$C(z,0)=\overline{C}(\lambda-\lambda X(z))\sum_{n=0}^{a-1}\sum_{m=a}^{b}P_{mn}(0)\,z^n \quad (3.1.9)$$

$$P_i(z,0)=\overline{S}(\lambda-\lambda X(z))\sum_{m=a}^{b}P_{mi}(0)z^m\quad a\leq i<b \quad (3.1.10)$$

$$z^b\,P_b\,(z,0)=\overline{S}(\lambda-\lambda X(z))\left\{\sum_{m=a}^{b-1}P_m(z,0)\right.$$
$$+P_b(z,0)-\sum_{m=a}^{b}\sum_{j=0}^{b-1}P_{mj}(0)z^n$$
$$\left.+\sum_{l=1}^{\infty}\left(Ql(z,0)-\sum_{j=0}^{b-1}Q_{lj}(0)z^j\right)\right\} \quad (3.1.11)$$

Now using (3.1.7) in (3.1.2), we get,

$$\overline{Q}_1\,(z,\theta)=\frac{\left[\overline{V}(\lambda-\lambda X(z))-\overline{V}(\theta)\right]\,C(z,0)}{(\theta-\lambda+\lambda X(z))} \quad (3.1.12)$$

From (3.1.8) and (3.1.3)

$$\overline{Q}_j(z,\theta)=\frac{\left[\overline{V}(\lambda-\lambda X(z))-\overline{V}(\theta)\right]\sum_{n=0}^{a-1}Q_{j-1n}(0)z^n}{(\theta-\lambda+\lambda X(z))}\quad j\geq 2 \tag{3.1.13}$$

From (3.1.9) and (3.1.4), we have

$$\overline{C}(z,\theta)=\frac{\left[\overline{C}(\lambda-\lambda X(z))-\overline{C}(\theta)\right]\sum_{n=0}^{a-1}\sum_{m=0}^{b}P_{mn}(0)z^n}{(\theta-\lambda-\lambda X(z))} \tag{3.1.14}$$

from the equation (3.1.10) and (3.1.5), we get,

$$\overline{P}_b(z,\theta)=\frac{\left[\overline{S}(\lambda-\lambda X(z))-\overline{S}(\theta)\right]\left[\sum_{m=0}^{b}P_{mi}(0)\sum_{l=1}^{\infty}Q_{li}(0)\right]}{(\theta-\lambda-\lambda X(z))}$$

$$a\leq i\leq b \tag{3.1.15}$$

Similarly, from (3.1.9) and (3.1.4), we have

$$\overline{P}_b(z,\theta)=\frac{\left[\overline{S}(\lambda-\lambda X(z))-\overline{S}(\theta)\right]f(z)}{(\theta-\lambda+\lambda X(z))(z^b-\overline{S})(\lambda-\lambda X(z))} \tag{3.1.16}$$

Where

$$f(z)=\sum_{m=a}^{b-1}P_m(z,0)-\sum_{m=a}^{b}\sum_{j=0}^{b-1}P_{mj}(0)z^j+\sum_{l=1}^{\infty}\left(Q_l(z,0)-\sum_{n=0}^{a-1}Q_{ln}(0)z^n\right)$$

Let P (z) be the PGF of the queue size at an arbitrary time epoch is the sum of PGF of queue size at service completion epoch, vacation completion epoch, closedown completion epoch, then

$$P(z)=\sum_{m=a}^{b-1}\overline{P}_m(z,0)+\overline{P}_b(z,0)+\overline{C}(z,0)+\sum_{l=1}^{\infty}\overline{Q}_l(z,0) \tag{3.1.17}$$

By substituting θ = 0 in the equation (3.1.12) – (3.1.16) then the equation (3.1.17) becomes

$$P(z)=\frac{\left[\overline{S}(\lambda-\lambda X(z))-1\right]\sum_{i=a}^{b-1}\left[\sum_{m=a}^{b}P_{mi}(0)+\sum_{l=1}^{\infty}Q_{li}(0)\right]}{(-\lambda+\lambda X(z))}$$
$$+\frac{\left[\overline{S}(\lambda-\lambda X(z))-1\right]f(z)}{(-\lambda+\lambda X(z))(z^b-\overline{S}(\lambda-\lambda X(z)))}$$
$$+\frac{\left[\overline{C}(\lambda-\lambda X(z))-1\right]\sum_{n=0}^{a-1}\sum_{m=a}^{b}P_{mn}(0)z^n}{(-\lambda+\lambda X(z))}$$
$$+\frac{\left[\overline{V}(\lambda-\lambda X(z))-1\right]\sum_{n=0}^{a-1}\left[C(z,0)+\sum_{j=1}^{\infty}Q_{jn}(0)z^n\right]}{(-\lambda+\lambda X(z))} \tag{3.1.18}$$

Let us define,

$$P_i=\sum_{m=a}^{b}P_{mi}(0)\ \text{and}\ q_j=\sum_{l=1}^{\infty}\overline{Q}_{lj}(0) \tag{3.1.19}$$

as the probabilities of the number of customers in queue at service completion epoch and vacation completion epoch respectively.

Let $c_i=p_i+q_i$,

By using (3.1.19), the equation (3.1.18) can be simplified as

$$P(z)=\Bigg\{\left(\overline{S}(\lambda-\lambda X(z))-1\right)\sum_{i=a}^{b-1}(z^b-z^i)c_i+(z^b-1)\left(\overline{V}(\lambda-\lambda X z)-1\right)\sum_{i=0}^{a-1}q_i z^i$$
$$X\ (\overline{S}(\lambda-\lambda X(z)-1)(1-\overline{C}(\lambda-\lambda X(z))\overline{V}(\lambda-\lambda X(z)))\sum_{i=0}^{a-1}p_i z^i$$
$$+(z^b-1)(\overline{V}(\lambda-\lambda X(z)))(\overline{C}(\lambda-\lambda X(z))-1)\sum_{i=0}^{a-1}p_i z^i\Bigg\}$$
$$/\ \left\{(-\lambda+\lambda X(z))(z^b-\overline{S}(\lambda-\lambda X(z)))\right\} \tag{3.1.20}$$

Which represents the PGF of number of customers in queue at an arbitrary time epoch.

3.2 Steady state condition :

The probability generating function has to satisfy p (1) = 1. In order to satisfy this condition applying L' Hospital's rule and evaluating $\underset{z\to j}{\mathrm{Lt}}\, P(z)$, then equating the expression to 1, we have,

$$E(S)\sum_{i=a}^{b-1}(b-i)\,p_i + b\big(E(C)+E(V)\big)\sum_{i=0}^{a-1}p_i$$

$$+bE(V)+\sum_{i=0}^{a-1}q_i-\lambda E(C)E(V)E(S)\sum_{i=0}^{a-1}p_i$$

$$=b-\lambda E(X)\,E(S)$$

since p_i, q_i are probabilities of 'i' customers being in the queue, it follows that left hand side of the above expression must be positive. Thus P (1) = 1 is satisfied iff $b-\lambda E(X)\,E(S)>0$, if $\rho=\lambda E(X)\,E(S)/b$ then $\rho<1$ is the condition to be satisfied for the existence of steady state for the model under consideration.

3.3 Computational aspects :

Equation (3.1.9) has b+a unknowns p_0, $p_1, p_2,\dots, p_{a-1}, q_0, q_1, q_2,\dots q_{a-1}, c_a, c_{a+1}, c_{b-1}$. We develop the following theorem to express q_i in terms of p_i, in such a way that numerator has only b constants. Now equation (3.1.20) gives the PGF of the number of customers involving only "b" unknowns. By Rouche's theorem of complex variables, it can be proved that $z^b-\overline{S}\,(\lambda-\lambda X(z))$ has b-1 zeros inside and one on the unit circle $|z|=1$. Since P (z) is analytic within and on the unit circle, the numerator must vanish at these points, which gives b equations in b unknowns. We can solve these equations by any suitable numerical techniques.

Theorem 1.

$$q_n=\sum_{i=0}^{n}K_i\,p_{n-i},\qquad n=0,1,2,3,\dots,a-1,$$

where

$$K_n=\frac{h_n+\sum_{i=1}^{n}\alpha_k K_{n-1}}{1-\alpha_0},\quad n=1,2,3,\dots,a-1$$

$$\text{with }K_0=\frac{\alpha_0\beta_0}{1-\alpha_0},\; h_n=\sum_{i=0}^{n}\alpha_0\beta_{n-i}$$

also α_i's and β_i's are the probabilities of the 'i' customers arrive during vacation time and closedown time respectively.

Proof. Using the equation (3.1.7) – (3.1.9), $\sum_{j=1}^{\infty}Q_j\,(z,0)$ simplifies to

$$\sum_{n=0}^{\infty}q_n z^n=\overline{V}\,(\lambda-\lambda X(z))\left[\overline{C}\,(\lambda-\lambda X(z))\sum_{n=0}^{a-1}p_n z^n+\sum_{n=0}^{a-1}q_n z^n\right]$$

$$=\left(\sum_{n=0}^{\infty}\alpha_n z^n\right)\left[\sum_{j=0}^{\infty}\beta_j\,z^j\sum_{n=0}^{a-1}p_n z^n+\sum_{n=0}^{N-1}q_n\,z^n\right]$$

$$=\sum_{j=0}^{n}\left(\sum_{i=0}^{n-j}\alpha_i\,\beta_{n-i-j}\right)p_j z^j+\sum_{i=0}^{n}\alpha_{n-i}\,q_i\,z^j$$

equation the coefficients of z^n on both sides of the above equation for n=0,1,2,.....,a-1 we have

$$q_n = \sum_{j=0}^{n}\left(\sum_{i=0}^{n-j}\alpha_i\beta_{n-i-j}\right)p_j + \sum_{i=0}^{n}\alpha_{n-i}q_i$$

on solving for q_n, we get

$$q_n = \frac{\left[\sum_{j=0}^{n}\left(\sum_{i=0}^{n-j}\alpha_i\beta_{n-i-j}\right)p_j + \sum_{i=0}^{n-1}\alpha_{n-i}q_i\right]}{(1-\alpha_0)}$$

Coefficient of p_n in q_n is $\frac{\alpha_0\beta_0}{1-\alpha_0} = K_0$.

Coefficient of p_n in q_{n-1} is $(h_1 + \alpha_1$ coefficient of p_{n-1} in $q_n)/(1-\alpha_0)$

$$= \frac{(h_1 + \alpha_1 K_0)}{(1-\alpha_0)} = K_1$$

hence by induction, the theorem follows.

Particular Case:

When closedown time is zero, the equation (3.1.17) reduces to

$$P(z) = \frac{\left\{\overline{S}(\lambda - \lambda X(z))\sum_{i=a}^{b-1}(z^b - z^i)P_i + (z^b - 1)(\overline{V}(\lambda - \lambda X(z)) - 1)\sum_{i=a}^{b-1} c_i z^i\right\}}{(-\lambda + \lambda X(z))\,(z^b - \overline{S}(\lambda - \lambda X(z)))} \quad (3.3.1)$$

Equation (3.3.1) gives the queue size distribution of $M^X/G(a,b)/1$ queueing system with multiple vacations. The result coincides with queue size distribution of Arumuganathan,[1] R.

3.4 Performance measures :

3.4.1 Expected queue length :

The expected queue length E (Q) at an arbitrary time epoch is obtained by differentiating P (z) at z = 1 and given by

$$E(Q) = \left[f_1(X,S)\sum_{i=a}^{b-1}(b-i)c_i + f_2(X,S)\sum_{n=a}^{b-1}(b(b-1) - i(i-1))c_i + f_3(X,S,V,C)\sum_{i=0}^{a-1}p_i + f_4(X,S,C,V)\sum_{i=0}^{a-1} ip_i + f_5(X,S,V)\sum_{i=0}^{a-1}(p_i + q_i) + f_6(X,V,S)\sum_{i=0}^{a-1} iq_i\right] \Big/ \left\{2(\lambda(E(X)(b - S1)))^2\right\} \quad (3.4.1.1)$$

The functions f_1, f_2, f_3, f_4, f_5 and f_6 are given by

$$f_1(X,S) = T2S1 - T1S1, \quad f_2(X,S) = T2S1,$$

$$f_3(X,S,C,V) = T2(S1-b)\,C1 + T1\,(b - 2S1)C2 + T1(b\,(b-1) - 2(b - S1)V1 - 2S2)C1,$$

$$f_3(X,S,C,V) = 2T1bV1 - 4T1S1C1, f5\,(X,V,S) = bT1(V2 + (b-1)V1 - bV1T2),$$

$$f_6(X,V,S) = 2bT1V1,$$

where

$S1 = \lambda E(X)E(S)$, $S2 = \lambda .X2.E(S) + \lambda^2 .X1^2 E(S^2)$,
$V1 = \lambda E(X)E(V)$,

$$V2 = \lambda .X2.E(V) + \lambda^2 .X1^2 .E(V^2),$$

$$T1 = \lambda X1(b - S1),$$

$$T2 = \lambda (X1(b(b-1) - S2 + X2(b - S1)),$$

$$C1 = \lambda E(X)E(C)),$$

$$C2 = \lambda .X2.E(C) + \lambda^2 .X_1^2 (C^2), \quad X1 = E(X),$$

$$X2 = X''(1).$$

3.4.2 Expected length of busy period :

Let B be the busy period random variable. We define the random variable J as

$$J \begin{cases} 0, & \text{if the server finds less than 'a' customers after the first service,} \\ 1, & \text{if the server finds at least 'a' customers after the first service.} \end{cases}$$

Now expected length of busy period is given by

$$E(B) = E(B/J=0)\,P(J=0) + E(B/J=1)\,P(J=1)$$
$$= E(S)\,P(J=0) + [E(S) + E(B)]\,P(J=1)$$

where E (S) is the mean service time.
Solving for E (B) we get,

$$E(B) = E(S)/P(J=0) = E(S) \Big/ \sum_{i=0}^{a-1} P_i \qquad (3.4.2.1)$$

3.4.3 Expected length of idle period :

Let I be the random variable. Then the expected length of idle period is given by, E (I)=E (I_1)+ E(C). Where I_1 is the random variable denoting the “Idle period due to multiple vacation process”, E (C) is the expected closedown time. We define the random variable U as,

$$J \begin{cases} 0, & \text{if the server finds at least 'a' customers after the first vacation,} \\ 1, & \text{if the server finds less than 'a' customers after the first vacation.} \end{cases}$$

Now the expected length of idle period of multiple vacations E(I_1) is given by

$$E(I_1) = E(I_1/U=0)\,P(U=0) + E(I_1/U=1)\,P(U=1)$$
$$= E\,(VP(U=0)) + [E(V) + E(I_1)]\,P(U=1).$$

Solving for E (I_1) we have,

$$E(I_1) = E(V)/P(u=0) \qquad (3.4.3.1)$$

From equation (3.1.7) we can get,

$$Q_{1n}(0) = \text{coefficient of } z^n \text{ in } Q_1(z, 0)$$

$$P(U=0) = 1 - \sum_{n=0}^{a-1} Q_{1n}(0) = 1 - \sum_{n=0}^{a-1} \sum_{i=0}^{n} \left\{ \sum_{j=0}^{n-1} \alpha_j \beta_{n-i-j} \right\} p_i, \qquad (3.4.3.2)$$

where α_i's, β_i's are the probabilities that ‘i’ customers during vacation and closedown time. Now the expected idle period E(I) is obtained as

$$E(I) = E(I_1) + E(C).$$

4.0 Cost Model :

Our queueing model has a bulk service rule with single server. The customers have to wait if sufficient batch quantity is not available, in such case the server will be on vacation or on closedown work. By considering this situation of customers waiting and the effective utilization of the operator and the machine, it is essential to have an optimal threshold value for a batch quantity. We develop a cost model through which the total costs involved in the system can be minimised.

We derive an expression for finding the total average cost with the following assumptions: Let C_s be the start up cost, C_h be the holding cost per customer. C_o be the operating cost per unit time, C_r be the reward per unit time due to vacation and C_u be the Closedown cost per unit time. The length of cycle is the sum of the idle period busy period, Now, the expected length of cycle; E (T_c), is obtained as,

$$E(T_c) = E(I) + E(B) = \frac{E(V)}{P(U=0)} + E(C) + E(S) \Big/ \sum_{i=0}^{a-1} P_i .$$

The total average cost per unit is given by,

Total average cost = start up cost per cycle

-Holding cost of number of customer in the queue

+operating cost* ρ+ Closedown time cost

- reward due to Vacation per cycle .

$$\text{Total average cost} = \left[C_s - C_r \cdot \frac{E(V)}{P(U=0)} + C_u . E(C) \right] \frac{1}{E(T_c)} + C_h . E(Q) + C_0 . \rho,$$

where ρ = AE(S)E(X)/b.

The significance of the cost model will be discussed with practical example in the next section.

5.0 Illustrative Example :

To illustrate the impact of proposed model we analyse the model numerically.

In the Globe Valve manufacturing industry, the components arrive in bulk from job shop follows Poisson distribution to CNC turning center, whose service follows an exponential distribution. At a time 10 components can be handled by CNC turning center. After the service. if the operator finds that the number of components available is less than the threshold value he performs closedown work.

The above system can be modeled as closedown times depending $M^X/G(a,b)/1$ queueing system with multiple vacations. If the CNC machine starts with threshold value l(the minimum capacity), the operating cost will be more and if we fix the threshold value 10 (the maximum capacity) due to that holding cost / vacation cost the total average cost will increases. We wish to obtain the optimum threshold value.

The unknown probabilities of the queue size distribution are computed using numerical techniques. Using Matlab software[9], the zeros of the function $z^b - \overline{S}(\lambda - \lambda X(z))$ are obtained and simultaneous equations are solved.

With the following parameters we analyse the above queueing system

(i) Service time distribution is K-Erlang distribution with k = 2, f-L = 5.

(ii) Batch arrival distribution is geometric.

(iii) Vacation time and Closedown time are exponential with parameters (α= 10, (β= 7 and

Startup cost:	Rs. 4.00
Holding cost per customer:	Rs. 0.50
Operating cost per unit time:	Rs. 5.00
Reward per unit time due to multiple vacations:	Rs. 1.00
Closedown time cost per unit time.	Rs. 0.25

The values of the above parameters are justified with test of goodness of fit[5].

The numerical results for various threshold values with b = 10 are presented in Table 1. A figure for threshold values Vs total average cost was also presented in Figure 1. From the Table 1 and Figure 1 it is clear that, for a CNC turning center with the capacity of 10 components at a time, the management has to fix, the threshold value as 3 to minimize the total average cost[7-10].

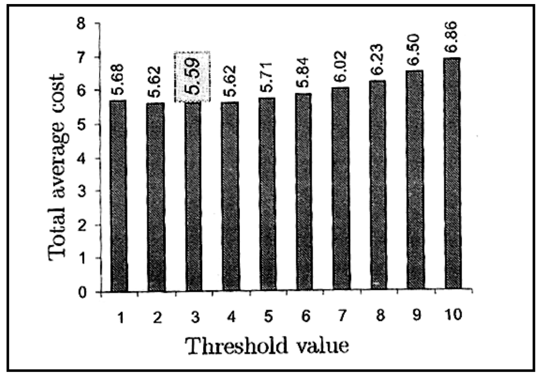

Figure 1. Threshold value Vs Total average cost.

Table 1. Performance measures and Total average cost

a	Unknown Probabilities					E(Q)	E(B)	E(I)	Total Average Cost
	1	3	5	7	9				
1	0.7813	0.2876	0.1193	0.0504	0.0216	1.47	0.512	0.2026	5.6797
2	0.5922	0.361	0.1392	0.0549	0.0221	1.5329	0.4748	0.2443	5.6229
3	0.4726	0.1838	0.182	0.068	0.0259	1.8418	0.4667	0.2694	5.5862
4	0.39	0.1533	0.2553	0.0921	0.0337	2.3017	0.4684	0.2818	5.6188
5	0.3289	0.1316	0.1172	0.1321	0.0471	2.8715	0.4738	0.2864	5.7091
6	0.2801	0.1158	0.1032	0.1967	0.069	3.5385	0.4804	0.2868	5.8438
7	0.2378	0.1041	0.0929	0.0871	0.1044	4.3107	0.4868	0.2851	6.0167
8	0.1967	0.0957	0.0858	0.0802	0.1624	5.2194	0.4925	0.2828	6.231
9	0.1509	0.0904	0.0815	0.0758	0.0727	6.3341	0.497	0.2807	6.5016
10	0.0913	0.0886	0.0805	0.0743	0.0707	7.8011	0.4998	0.2792	6.8642

α– Threshold value, E (Q)-Expected queue length, E (B) -Expected length of busy period, E (I)-Expected idle time

6.0 Conclusion

In this paper a closedown times depending a $M^X/G(a,b)/1$ queineg system with multiple vacations is considered. Probability generating function of queue size at an arbitrary time epoch and various performance measures are obtained. We analysed the impact of cost model numerically to a practical situation for decision making process. It is left for future research that effect of considering set up period. Cost model analysed in this paper can be used to analyse a similar model with setup period and N-Policy. Further the PGF of queue size can be decomposed at various epochs in to several factors.

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