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A generalization of fuzzy alpha-boundary

K. BAGEERATHI[1] and T. THANGAM[2]

[1,2]Department of Mathematics, Govindammal Aditanar College for Women,
Tiruchendur-628215 (India)
e-mail: sivarathi_2006@yahoo.in, thangammathavan@yahoo.com

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Abstract

The author has recently introduced the concepts of fuzzy **C** – boundary[3], fuzzy **C** -semi-boundary[6] and fuzzy **C** -pre boundary[7] where **C** :$[0, 1]\rightarrow[0, 1]$ is a function. The purpose of this paper is to introduce the concept of fuzzy **C**-alpha boundary and investigate some of their basic properties of a fuzzy topological space.

Key words: Fuzzy **C** - alpha boundary, fuzzy **C** - α- closed sets and fuzzy topology.

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1. Introduction

The concept of complement function is used to define a fuzzy closed subset of a fuzzy topological space. That is a fuzzy subset λ is fuzzy closed if the standard complement $1-\lambda = \lambda'$ is fuzzy open. Here the standard complement is obtained by using the function **C** : $[0, 1]\rightarrow [0, 1]$ defined by **C** $(x) = 1-x$, for all $x \in[0, 1]$. Several fuzzy topologists used this type of complement while extending the concepts in general topological spaces to fuzzy topological spaces. But there are other complements in the fuzzy literature[10]. This motivated the second and third authors to introduce the concepts of fuzzy **C** -closed sets[2] and fuzzy **C** - α-closed sets[5] in fuzzy topological spaces. In this paper, we introduce the concept of fuzzy **C** - alpha boundary by using the arbitrary complement function **C** and fuzzy **C** -α-closure operator.

For the basic concepts and notations, one can refer Chang[8]. The concepts that are needed in this paper are discussed in the second section. The concepts of fuzzy**C** -α-interior and fuzzy **C** -α-closure are introduced in the third section. The section four is dealt with

Corresponding author : K.Bageerathi[1],
[1]Department of Mathematics, Govindammal Aditanar College for Women, Tiruchendur-628215, India.
e-mail: sivarathi_2006@yahoo.in

the concept of fuzzy **C** -ahpha-boundary.

2. Preliminaries :

Throughout this paper (X,τ) denotes a fuzzy topological space in the sense of Chang. Let $\mathbf{C} : [0, 1]\to[0, 1]$ be a complement function. If λ is a fuzzy subset of (X,τ) then the complement $\mathbf{C}\,\lambda$ of a fuzzy subset λ is defined by $\mathbf{C}\,\lambda(x) = \mathbf{C}\,(\lambda(x))$ for all $x\in X$. A complement function **C** is said to satisfy

(i) the boundary condition if $\mathbf{C}(0) = 1$ and $\mathbf{C}(1) = 0$,

(ii) monotonic condition if $x \le y \Rightarrow \mathbf{C}\,(x) \ge \mathbf{C}\,(y)$, for all $x, y \in [0, 1]$,

(iii) involutive condition if $\mathbf{C}\,(\mathbf{C}\,(x)) = x$, for all $x \in[0, 1]$.

The properties of fuzzy complement function **C** and $\mathbf{C}\,\lambda$ are given in George Klir[8] and Bageerathi *et al*.[2]. The following lemma will be useful in sequel.

Lemma 2.1[2]

Let $\mathbf{C} : [0, 1]\to [0, 1]$ be a complement function that satisfies the monotonic and involutive conditions. Then for any family $\{\lambda_\alpha: \alpha\in\Delta\}$ of fuzzy subsets of X, we have

(i) $\mathbf{C}\,(\sup\{\lambda_\alpha(x):\alpha\in\Delta\})=\inf\{\mathbf{C}\,(\lambda_\alpha(x)):\alpha\in\Delta\}= \inf\{(\mathbf{C}\,\lambda_\alpha(x)): \alpha\in\Delta\}$ and

(ii) $\mathbf{C}\,(\inf\{\lambda_\alpha(x): \alpha\in\Delta\}) = \sup\{\mathbf{C}\,(\lambda_\alpha(x)): \alpha\in\Delta\}= \sup\{(\mathbf{C}\,\lambda_\alpha(x)): \alpha\in\Delta\}$ for $x\in X$.

Definition 2.2[2]

A fuzzy subset λ of X is fuzzy **C** -closed in (X,τ) if $\mathbf{C}\,\lambda$ is fuzzy open in (X,τ). The fuzzy **C** -closure of λ is defined as the intersection of all fuzzy **C** -closed sets μ containing λ. The fuzzy **C** -closure of λ is denoted by $Cl_{\mathbf{C}}\lambda$ that is equal to $\wedge\{\mu: \mu \ge \lambda, \mathbf{C}\,\mu\in\tau\}$.

Lemma 2.3[2]

If the complement function **C** satisfies the monotonic and involutive conditions, then for any fuzzy subset λ of X,

(i) $\mathbf{C}\,(Int\,\lambda) = Cl_{\mathbf{C}}(\mathbf{C}\,\lambda)$ and $\mathbf{C}\,(Cl_{\mathbf{C}}\lambda) = Int\,(\mathbf{C}\,\lambda)$.

(ii) $\lambda \le Cl_{\mathbf{C}}\,\lambda$,

(iii) λ is fuzzy **C** -closed $\Leftrightarrow$ $Cl_{\mathbf{C}}\,\lambda= \lambda$,

(iv) $Cl_{\mathbf{C}}\,(Cl_{\mathbf{C}}\,\lambda) = Cl_{\mathbf{C}}\,\lambda$,

(v) If $\lambda \le \mu$ then $Cl_{\mathbf{C}}\lambda \le Cl_{\mathbf{C}}\,\mu$,

(vi) $Cl_{\mathbf{C}}\,(\lambda\vee \mu) = Cl_{\mathbf{C}}\,\lambda \vee Cl_{\mathbf{C}}\,\mu$,

(vii) $Cl_{\mathbf{C}}\,(\lambda\wedge \mu) \le Cl_{\mathbf{C}}\,\lambda \wedge Cl_{\mathbf{C}}\,\mu$.

(viii) For any family $\{\lambda_\alpha\}$ of fuzzy sub sets of a fuzzy topological space we have $\vee\, Cl_{\mathbf{C}}\,\lambda_\alpha \le Cl_{\mathbf{C}}\,(\vee\lambda_\alpha)$ and $Cl_{\mathbf{C}}\,(\wedge_{\alpha}) \le \wedge\, Cl_{\mathbf{C}}\,\lambda_\alpha$.

Lemma 2.4[2]

Let (X,τ) be a fuzzy topological space. Let **C** be a complement function that satisfies the boundary, monotonic and involutive conditions. Then the following conditions hold.

(i) 0 and 1 are fuzzy **C** -closed sets,

(ii) arbitrary intersection of fuzzy **C** -closed sets is fuzzy **C** -closed and

(iii) finite union of fuzzy **C** -closed sets is fuzzy **C** -closed.

(iv) for any family $\{\lambda_\alpha : \alpha\in \Delta\}$ of fuzzy subsets of X. we have

$\mathbf{C}\,(\vee\{\lambda_\alpha:\ \alpha\in\Delta\}) = \wedge\{\mathbf{C}\,\lambda_\alpha : \alpha\in\Delta\}$ and

$\mathbf{C}\,(\wedge\{\lambda_\alpha:\ \alpha\in\Delta\}) = \vee\{\mathbf{C}\,\lambda_\alpha : \alpha\in\Delta\}$

Definition 2.5 [Definition 2.15,[3]]

A fuzzy topological space (X, τ) is **C** -product related to another fuzzy topological space (Y, σ) if for any fuzzy subset ν of X and ζ of Y, whenever $\mathbf{C}\,\lambda \not\ge \nu$ and $\mathbf{C}\,\mu\ \ \zeta$ imply $\mathbf{C}\,\lambda \times 1 \vee 1 \times \mathbf{C}\,\mu \ge \nu \times \zeta$, where $\lambda \in \tau$ and $\mu \in\sigma$, there exist $\lambda_1 \in\tau$ and $\mu_1\in\sigma$ such that $\mathbf{C}\,\lambda_1$

$\geq \nu$ or $\mathbf{C}\,\mu_1 \geq \zeta$ and $\mathbf{C}\,\lambda_1 \times 1 \vee 1 \times \mathbf{C}\,\mu_1 = \mathbf{C}\,\lambda \times 1 \vee 1 \times \mathbf{C}\,\mu$.

Lemma 2.6 [Theorem 2.19,[3]]

Let (X, τ) and (Y, σ) be $\mathbf{C}$ -product related fuzzy topological spaces. Then for a fuzzy subset λ of X and a fuzzy subset μ of Y, $Cl_{\mathbf{C}}(\lambda \ \mu) = Cl_{\mathbf{C}}\,\lambda \times Cl_{\mathbf{C}}\,\mu$.

Definition 2.7 [Definition 3.1, [4]]

Let (X,τ) be a fuzzy topological space and $\mathbf{C}$ be a complement function. Then a fuzzy subset λ of X is fuzzy $\mathbf{C}$-α-open if $\lambda \leq Int\, Cl_{C}\, Int\lambda$.

Definition 2.7 [Definition 3.1, [4]]]

Let (X,τ) be a fuzzy topological space and $\mathbf{C}$ be a complement function. Then a fuzzy subset λ of X is fuzzy $\mathbf{C}$-α-open if $\lambda \leq Int\, Cl_{\mathbf{C}}\ Int\lambda$.

Lemma 2.8 [4,5]

Let (X, τ) be a fuzzy topological space and let $\mathbf{C}$ be a complement function that satisfies the monotonic and involutive properties. Then a fuzzy set λ of a fuzzy topological space (X,τ) is

(i) fuzzy $\mathbf{C}$ - α- open if and only if $\lambda \leq Int\,(Cl_{\mathbf{C}}\,\lambda)$.

(ii) fuzzy $\mathbf{C}$ -α-closed if and only if $\mathbf{C}\,\lambda$ is fuzzy $\mathbf{C}$ -α-open.

(iii) the arbitrary union of fuzzy $\mathbf{C}$ -α-open sets is fuzzy $\mathbf{C}$ -α-open.

Lemma 2.9[1]

If $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the fuzzy subsets of X then

$(\lambda_1 \wedge \lambda_2) \times (\lambda_3 \wedge \lambda_4) = (\lambda_1 \times \lambda_4) \wedge (\lambda_2 \times \lambda_3)$.

Lemma 2.10 [Lemma 5.1,[2]]

Suppose f is a function from X to Y. Then $f^{-1}(\mathbf{C}\ \mu) = \mathbf{C}\,(f^{-1}(\mu))$ for any fuzzy subset μ of Y.

Definition 2.11[7]

If λ is a fuzzy subset of X and μ is a fuzzy subset of Y, then $\lambda \times \mu$ is a fuzzy subset of $X \times Y$, defined by $(\lambda \times \mu)(x, y) = \min\{\lambda(x), \mu(y)\}$ for each $(x, y) \in X \times Y$.

Lemma 2.12 [Lemma 2.1,[1]]

Let $f : X \rightarrow Y$ be a function. If $\{\lambda_\alpha\}$ a family of fuzzy subsets of Y, then

(i) $f^{-1}(\vee\lambda_\alpha) = \vee f^{-1}(\lambda_\alpha)$ and

(ii) $f^{-1}(\wedge\lambda_\alpha) = \wedge f^{-1}(\lambda_\alpha)$.

Lemma 2.13 [Lemma 2.2,[1]]

If λ is a fuzzy subset of X and μ is a fuzzy subset of Y, then $\mathbf{C}\,(\lambda\times\mu) = \mathbf{C}\,\lambda\, 1 \vee 1 \times \mathbf{C}\,\mu$.

3. Fuzzy $\mathbf{C}$ -α- interior and fuzzy $\mathbf{C}$ -α- closure :

In this section, we define the concepts of fuzzy $\mathbf{C}$ -α- interior and fuzzy $\mathbf{C}$ -α- closure operators and investigate some of their basic properties.

Definition 3.1

Let (X,τ) be a fuzzy topological space and $\mathbf{C}$ be a complement function. Then for a fuzzy subset λ of X, the fuzzy $\mathbf{C}$ - α- interior of λ (briefly α-$Int_{\mathbf{C}}\,\lambda$), is the union of all fuzzy $\mathbf{C}$ - α- open sets of X contained in λ. That is,

α-$Int_{\mathbf{C}}\,(\lambda) = \vee\,\{\mu\colon \mu \leq \lambda,\ \mu$ is fuzzy $\mathbf{C}$ - α-open$\}$.

Proposition 3.2

Let (X, τ) be a fuzzy topological space and let **C** be a complement function that satisfies the monotonic and involvtive conditions. Then for any fuzzy subsets λ and μ of a fuzzy topological space X, we have

(i) $\alpha\text{-}Int_{\mathbf{C}}\lambda \leq \lambda$,

(ii) λ is fuzzy**C** - α- open $\Leftrightarrow \alpha\text{-}Int_{\mathbf{C}}\lambda = \lambda$,

(iii) $\alpha\text{-}Int_{\mathbf{C}}\,(pInt_{\mathbf{C}}\lambda) = \alpha\text{-}Int_{\mathbf{C}}\lambda$,

(iv) If $\lambda \leq \mu$ then $\alpha\text{-}Int_{\mathbf{C}}\lambda \leq \alpha\text{-}Int_{\mathbf{C}}\,\mu$.

Proof.

The proof for (i) follows from Definition 3.1. Let λ be fuzzy **C** - α- open. Since $\lambda \leq \lambda$, by using Definition 3.1, $\lambda \leq \alpha\text{-}Int_{\mathbf{C}}\lambda$. By using (i), we get $\alpha\text{-}Int_{\mathbf{C}}\lambda = \lambda$. Conversely we assume that $\alpha\text{-}Int_{\mathbf{C}}\lambda = \lambda$. By using Definition 3.1, λ is fuzzy **C** - α- open. Thus (ii) is proved. By using (ii), we get $\alpha\text{-}Int_{\mathbf{C}}\,(\alpha\text{-}Int_{\mathbf{C}}\lambda) = \alpha\text{-}Int_{\mathbf{C}}\lambda$. This proves (iii). Since $\leq \mu$, by using (i), $\alpha\text{-}Int_{\mathbf{C}}\lambda \leq \lambda \leq \mu$. This implies that $\alpha\text{-}Int_{\mathbf{C}}\,(\alpha\text{-}Int_{\mathbf{C}}\lambda) \leq \alpha\text{-}Int_{\mathbf{C}}\,\mu$. By using (iii), we get $\alpha\text{-}Int_{\mathbf{C}}\,\lambda \leq \alpha\text{-}Int_{\mathbf{C}}\,\mu$. This proves (iv).

Proposition 3.3

Let (X, τ) be a fuzzy topological space and let**C** be a complement function that satisfies the monotonic and involutive conditions. Then for any two fuzzy subsets λ and μ of a fuzzy topological space, we have (i) $\alpha\text{-}Int_{\mathbf{C}}\,(\lambda\vee\mu) \geq \alpha\text{-}Int_{\mathbf{C}}\,\lambda\vee\,\alpha\text{-}Int_{\mathbf{C}}\,\mu$ and (ii) $\alpha\text{-}Int_{\mathbf{C}}\,(\lambda\wedge\mu) \leq \alpha\text{-}Int_{\mathbf{C}}\,\lambda\wedge\alpha\text{-}Int_{\mathbf{C}}\,\mu$.

Proof.

Since $\lambda \leq \lambda \vee \mu$ and $\mu \leq \lambda\vee\mu$, by using Proposition 3.2(iv), we get $\alpha\text{-}Int_{\mathbf{C}}\,\lambda \leq \alpha\text{-}Int_{\mathbf{C}}\,(\lambda\vee\mu)$ and $\alpha\text{-}Int_{\mathbf{C}}\,\mu \leq \alpha\text{-}Int_{\mathbf{C}}\,(\lambda\vee\mu)$. This implies that $\alpha\text{-}Int_{\mathbf{C}}\,\lambda\vee\,\alpha\text{-}Int_{\mathbf{C}}\,\mu \leq \alpha\text{-}Int_{\mathbf{C}}\,(\lambda\vee\mu)$.

Since $\lambda \wedge\ \leq \lambda$ and $\lambda \wedge \mu \leq \mu$, by using Proposition 3.2(iv), we get $\alpha\text{-}Int_{\mathbf{C}}\,(\lambda\wedge\mu)\ \ \alpha\text{-}Int_{\mathbf{C}}\,\lambda$ and $\alpha\text{-}Int_{\mathbf{C}}\,(\lambda\wedge\mu) \leq \alpha\text{-}Int_{\mathbf{C}}\,\mu$. This implies that $\alpha\text{-}Int_{\mathbf{C}}\,(\lambda\wedge\mu) \leq \alpha\text{-}Int_{\mathbf{C}}\,\lambda \wedge \alpha\text{-}Int_{\mathbf{C}}\,\mu$.

Definition 3.4

Let (X,τ) be a fuzzy topological space. Then for a fuzzy subset λ of X, the fuzzy **C** - α- closure of λ (briefly $\alpha\text{-}Cl_{\mathbf{C}}\lambda$), is the intersection of all fuzzy **C** - α- closed sets containing λ. That is $\alpha\text{-}Cl_{\mathbf{C}}\lambda = \wedge\{\mu\text{: } \mu \geq \lambda, \mu$ is fuzzy **C** - α-closed$\}$.

The concepts of "fuzzy **C**-α- closure" and "fuzzy α-closure" are identical if **C** is the standard complement function.

Proposition 3.5

If the complement functions**C** satisfies the monotonic and involutive conditions. Then for any fuzzy subset λ of X, (i)**C** $(\alpha\text{-}Int_{\mathbf{C}}\lambda) = \alpha\text{-}Cl_{\mathbf{C}}\,(\mathbf{C}\,\lambda)$ and

(ii)**C** $(\alpha\text{-}Cl_{\mathbf{C}}\lambda) = \alpha\text{-}Int_{\mathbf{C}}\,(\mathbf{C}\,\lambda)$, where $\alpha\text{-}Int_{\mathbf{C}}\lambda$ is the union of all fuzzy**C** -α-open sets contained in λ.

Proof.

By using Definition 3.1, $\alpha\text{-}Int_{\mathbf{C}}\lambda = \vee\{\mu\text{: } \leq \lambda,\ \mu$ is fuzzy **C** - α- open$\}$. Taking complement on both sides, we get **C** $(\alpha\text{-}Int_{\mathbf{C}}(\lambda)(x)) = \mathbf{C}\,(\sup\{\mu(x)\text{: } \mu(x) \leq \lambda(x), \mu$ is fuzzy **C** -α- open$\})$. Since **C** satisfies the monotonic and involutive conditions, by using Lemma 2.1, **C** $(\alpha\text{-}Int_{\mathbf{C}}(\lambda)(x)) = \inf\{\mathbf{C}\,(\mu(x))\text{: } \mu(x) \leq \lambda(x)$, μ is fuzzy**C** - α- open$\}$. This implies that **C** $(\alpha\text{-}Int_{\mathbf{C}}(\lambda)(x)) = \inf\{\mathbf{C}\,\mu(x))\text{:}\mathbf{C}\,\mu(x) \geq \mathbf{C}\,\lambda(x)$, μ is fuzzy **C** - α- open$\}$. By using Lemma 2.8,

$\mathbf{C}\,\mu$ is fuzzy $\mathbf{C}$ - α- closed, by replacing $\mathbf{C}\,\mu$ by η, we see that $\mathbf{C}\,(\text{-}Int_{\mathbf{C}}(\lambda)(x)) = \inf\{\eta(x): \eta(x) \geq \mathbf{C}\,\lambda(x), \mathbf{C}\,\eta$ is fuzzy $\mathbf{C}$ - α- open$\}$. By Definition 3.4, $\mathbf{C}\,(\alpha\text{-}Int_{\mathbf{C}}(\lambda)(x)) = \alpha\text{-}Cl_{\mathbf{C}}(\mathbf{C}\,\lambda)(x)$. This proves that $\mathbf{C}\,(\alpha\text{-}Int_{\mathbf{C}}\,\lambda) = \alpha\text{-}Cl_{\mathbf{C}}(\mathbf{C}\,\lambda)$.

By using Definition 3.4, $\alpha\text{-}Cl_{\mathbf{C}}\lambda = \wedge\{\mu: \leq\mu, \mu$ is fuzzy $\mathbf{C}$ - α- closed$\}$. Taking complement on both sides, we get $\mathbf{C}\,(\alpha\text{-}Cl_{\mathbf{C}}\,\lambda(x)) = \mathbf{C}\,(\inf\{\mu(x): \mu(x)\ \lambda(x), \mu$ is fuzzy $\mathbf{C}$ -α-closed$\})$. Since $\mathbf{C}$ satisfies the monotonic and involutive conditions, by using Lemma 2.1, $\mathbf{C}\,(\text{-}Cl_{\mathbf{C}}\lambda(x)) = \sup\{\mathbf{C}\,(\mu(x)): \mu(x) \geq \lambda(x), \mu$ is fuzzy $\mathbf{C}$ - α- closed$\}$. That implies $\mathbf{C}\,(\alpha\text{-}Cl_{\mathbf{C}}\lambda(x)) = \sup\{\mathbf{C}\,\mu(x): \mathbf{C}\,\mu(x)) \leq \mathbf{C}\,\lambda(x), \mu$ is fuzzy $\mathbf{C}$ - α- closed$\}$. By using Lemma 2.8, $\mathbf{C}\,\mu$ is fuzzy $\mathbf{C}$ -α-open, by replacing $\mathbf{C}\,\mu$ by η, we see that $\mathbf{C}\,(\alpha\text{-}Cl_{\mathbf{C}}\,(x)) = \sup\{\eta(x): \eta(x) \leq \mathbf{C}\,\lambda(x), \eta$ is fuzzy $\mathbf{C}$ -α- open$\}$. By using Definition 3.1, $\mathbf{C}\,(\alpha\text{-}Cl_{\mathbf{C}}\,\lambda(x)) = \alpha\text{-}Int_{\mathbf{C}}(\mathbf{C}\,\lambda)(x)$. This proves $\mathbf{C}\,(\alpha\text{-}Cl_{\mathbf{C}}(\lambda)) = \alpha\text{-}Int_{\mathbf{C}}(\mathbf{C}\,\lambda)$.

Proposition 3.6

Let (X, τ) be a fuzzy topological space and let $\mathbf{C}$ be a complement function that satisfies the monotonic and involutive conditions. Then for the fuzzy subsets λ and μ of a fuzzy topological space X, we have

(i) $\lambda \leq \alpha\text{-}Cl_{\mathbf{C}}\,\lambda$,

(ii) λ is fuzzy $\mathbf{C}$ - α- closed $\Leftrightarrow \alpha\text{-}Cl_{\mathbf{C}}\,\lambda = \lambda$,

(iii) $\alpha\text{-}Cl_{\mathbf{C}}\,(\alpha\text{-}Cl_{\mathbf{C}}\,\lambda) = \alpha\text{-}Cl_{\mathbf{C}}\,\lambda$,

(iv) If $\lambda \leq \mu$ then $\alpha\text{-}Cl_{\mathbf{C}}\,\lambda \leq \alpha\text{-}Cl_{\mathbf{C}}\,\mu$.

Proof.

The proof for (i) follows from $\alpha\text{-}Cl_{\mathbf{C}}\,\lambda = \inf\{\mu: \mu \geq \lambda$, is fuzzy $\mathbf{C}$ - α- closed$\}$.
Let λ be fuzzy $\mathbf{C}$ - α-closed. Since $\mathbf{C}$ satisfies the monotonic and involvtive conditions. Then by using Lemma 2.8, $\mathbf{C}\,\lambda$ is fuzzy $\mathbf{C}$ - α- open. By using Proposition 3.2(ii), $\alpha\text{-}Int_{\mathbf{C}}(\mathbf{C}\,\lambda) = \mathbf{C}\lambda$. By using Proposition 3.5, $\mathbf{C}\,(\alpha\text{-}Cl_{\mathbf{C}}\,\lambda) = \mathbf{C}\,\lambda$. Taking complement on both sides, we get $\mathbf{C}\,(\mathbf{C}\,(\alpha\text{-}Cl_{\mathbf{C}}\,\lambda)) = \mathbf{C}\,(\mathbf{C}\,\lambda)$. Since the complement function $\mathbf{C}$ satisfies the involutive condition, $\alpha\text{-}Cl_{\mathbf{C}}\,\lambda = \lambda$.

Conversely, we assume that $\alpha\text{-}Cl_{\mathbf{C}}\,\lambda = \lambda$. Taking complement on both sides, we get $\mathbf{C}\,(\alpha\text{-}Cl_{\mathbf{C}}\,\lambda) = \mathbf{C}\,\lambda$. By using Proposition 3.5, $\alpha\text{-}Int_{\mathbf{C}}\mathbf{C}\,\lambda = \mathbf{C}\,\lambda$. By using Proposition 3.2 (ii), $\mathbf{C}\,\lambda$ is fuzzy $\mathbf{C}$ - α-open. Again by using Lemma 2.8, λ is fuzzy $\mathbf{C}$ - α- closed. Thus (ii) proved.

By using Proposition 3.5, $\mathbf{C}\,(\alpha\text{-}Cl_{\mathbf{C}}\,\lambda) = \alpha\text{-}Int_{\mathbf{C}}(\mathbf{C}\,\lambda)$. This implies that $\mathbf{C}\,(\alpha\text{-}Cl_{\mathbf{C}}\,\lambda)$ is fuzzy $\mathbf{C}$ - α- open. By using Lemma 2.8, $\alpha\text{-}Cl_{\mathbf{C}}\,\lambda$ is fuzzy $\mathbf{C}$ -α- closed. By applying (ii), we have $\alpha\text{-}Cl_{\mathbf{C}}\,(\alpha\text{-}Cl_{\mathbf{C}}\,\lambda) = \alpha\text{-}Cl_{\mathbf{C}}\,\lambda$. This proves (iii).

Suppose $\lambda \leq \mu$. Since $\mathbf{C}$ satisfies the monotonic condition, $\mathbf{C}\,\lambda \geq \mathbf{C}\,\mu$, that implies $\alpha\text{-}Int_{\mathbf{C}}\,\mathbf{C}\,\lambda \geq \alpha\text{-}Int_{\mathbf{C}}\,\mathbf{C}\mu$. Taking complement on both sides, $\mathbf{C}\,(\alpha\text{-}Int_{\mathbf{C}}\,\mathbf{C}\,\lambda) \leq \mathbf{C}\,(\alpha\text{-}Int_{\mathbf{C}}\mathbf{C}\,\mu)$. Then by using Proposition 3.5, $\alpha\text{-}Cl_{\mathbf{C}}\,\lambda \leq \alpha\text{-}Cl_{\mathbf{C}}\,\mu$. This proves (iv).

Proposition 3.7

Let (X, τ) be a fuzzy topological space and let $\mathbf{C}$ be a complement function that satisfies the monotonic and involutive conditions. Then for any two fuzzy subsets λ and μ of a fuzzy topological space, we have (i) $\alpha\text{-}Cl_{\mathbf{C}}\,(\lambda \vee \mu) = \alpha\text{-}Cl_{\mathbf{C}}\,\lambda \vee \alpha\text{-}Cl_{\mathbf{C}}\,\mu$ and
(ii) $\alpha\text{-}Cl_{\mathbf{C}}\,(\lambda \wedge \mu) \leq \alpha\text{-}Cl_{\mathbf{C}}\,\lambda \wedge \alpha\text{-}Cl_{\mathbf{C}}\,\mu$.

Proof.

Since $\mathbf{C}$ satisfies the involutive condition, $\alpha\text{-}Cl_{\mathbf{C}}(\lambda\vee\mu)=\alpha\text{-}Cl_{\mathbf{C}}(\mathbf{C}(\mathbf{C}(\lambda\vee\mu)))$. Since $\mathbf{C}$ satisfies the monotonic and involutive conditions, by using Proposition 3.5, $\alpha\text{-}Cl_{\mathbf{C}}(\lambda\vee\mu)=\mathbf{C}(\alpha\text{-}Int_{\mathbf{C}}(\mathbf{C}(\lambda\vee\mu)))$. Using Lemma 2.10, $\alpha\text{-}Cl_{\mathbf{C}}(\lambda\vee\mu)=\mathbf{C}(\alpha\text{-}Int_{\mathbf{C}}(\mathbf{C}\lambda\wedge\mathbf{C}\mu))$. Again by Lemma 2.4, $\alpha\text{-}Cl_{\mathbf{C}}(\lambda\vee\mu)\leq\mathbf{C}((\alpha\text{-}Int_{\mathbf{C}}\mathbf{C}\lambda)\wedge(\alpha\text{-}Int_{\mathbf{C}}\mathbf{C}\mu))=\mathbf{C}(\alpha\text{-}Int_{\mathbf{C}}\mathbf{C}\lambda)\vee\mathbf{C}(\alpha\text{-}Int_{\mathbf{C}}\mathbf{C}\mu)$. By using Proposition 3.5, $\alpha\text{-}Cl_{\mathbf{C}}(\lambda\vee\mu)\leq\alpha\text{-}Cl_{\mathbf{C}}\lambda\alpha\text{-}Cl_{\mathbf{C}}\mu$. Also $\alpha\text{-}Cl_{\mathbf{C}}(\lambda)\leq\alpha\text{-}Cl_{\mathbf{C}}(\lambda\vee\mu)$ and $\alpha\text{-}Cl_{\mathbf{C}}(\mu)\leq\alpha\text{-}Cl_{\mathbf{C}}(\lambda\vee\mu)$ that implies $\alpha\text{-}Cl_{\mathbf{C}}(\lambda)\vee\alpha\text{-}Cl_{\mathbf{C}}(\mu)\leq\alpha\text{-}Cl_{\mathbf{C}}(\lambda\vee\mu)$. It follows that $\alpha\text{-}Cl_{\mathbf{C}}(\lambda\vee\mu)=\alpha\text{-}Cl_{\mathbf{C}}\lambda\vee\alpha\text{-}Cl_{\mathbf{C}}\mu$.

Since $\alpha\text{-}Cl_{\mathbf{C}}(\lambda\wedge\mu)\leq\alpha\text{-}Cl_{\mathbf{C}}\lambda$ and $\alpha\text{-}Cl_{\mathbf{C}}(\lambda\wedge\mu)\leq\alpha\text{-}Cl_{\mathbf{C}}\mu$, it follows that $\alpha\text{-}Cl_{\mathbf{C}}(\lambda\wedge\mu)\leq\alpha\text{-}Cl_{\mathbf{C}}\lambda\wedge\alpha\text{-}Cl_{\mathbf{C}}\mu$.

Proposition 3.8

Let $\mathbf{C}$ be a complement function that satisfies the monotonic and involutive conditions. Then for any family $\{\lambda_\alpha\}$ of fuzzy subsets of a fuzzy topological space, we have (i) $\vee(\alpha\text{-}Cl_{\mathbf{C}}\lambda_\alpha)\leq\alpha\text{-}Cl_{\mathbf{C}}(\vee\lambda_\alpha)$ and (ii) $\alpha\text{-}Cl_{\mathbf{C}}(\wedge\lambda_\alpha)\leq\wedge(\alpha\text{-}Cl_{\mathbf{C}}\lambda_\alpha)$

Proof.

For every β, $\lambda_\beta\leq\vee\lambda_\alpha\leq pCl_{\mathbf{C}}(\vee\lambda_\alpha)$. By using Proposition 3.6 (iv), $pCl_{\mathbf{C}}\lambda_\beta\leq pCl_{\mathbf{C}}(\vee\lambda_\alpha)$ for every β. This implies that $\vee pCl_{\mathbf{C}}\lambda_\beta\leq pCl_{\mathbf{C}}(\vee\lambda_\alpha)$. This proves (i). Now $\wedge\lambda_\alpha\leq\lambda_\beta$ for every β. Again using Proposition 3.6(iv), we get $\alpha\text{-}Cl_{\mathbf{C}}(\wedge\lambda_\alpha)\leq\alpha\text{-}Cl_{\mathbf{C}}\lambda_\beta$. This implies that $\alpha\text{-}Cl_{\mathbf{C}}(\wedge\lambda_\alpha)\leq\wedge\alpha\text{-}Cl_{\mathbf{C}}\lambda_\alpha$. This proves (ii).

4. *Fuzzy* C *-α-boundary :*

In this section, the concept of fuzzy $\mathbf{C}$-α-boundary is introduced and its properties are discussed[9].

Definition 4.1

Let λ be a fuzzy subset of a fuzzy topological space X and let $\mathbf{C}$ be a complement function. Then the fuzzy $\mathbf{C}$-α-boundary of λ is defined as

$\alpha\text{-}Bd_{\mathbf{C}}\lambda=\alpha\text{-}Cl_{\mathbf{C}}\lambda\wedge\alpha\text{-}Cl_{\mathbf{C}}(\mathbf{C}\lambda)$.

Since the arbitrary intersection of fuzzy $\mathbf{C}$-α-closed sets is fuzzy $\mathbf{C}$-α-closed, $\alpha\text{-}Bd_{\mathbf{C}}\lambda$ is fuzzy $\mathbf{C}$-α-closed.

We identify $\alpha\text{-}Bd_{\mathbf{C}}\lambda$ with $\alpha\text{-}Bd(\lambda)$ when $\mathbf{C}(x)=1-x$, the usual complement function.

Proposition 4.2

Let (X,τ) be a fuzzy topological space and $\mathbf{C}$ be a complement function that satisfies the involutive condition. Then for any fuzzy subset λ of X, $\alpha\text{-}Bd_{\mathbf{C}}\lambda=\alpha\text{-}Bd_{\mathbf{C}}(\mathbf{C}\lambda)$.

Proof.

By using Definition 4.1, $\alpha\text{-}Bd_{\mathbf{C}}\lambda=\alpha\text{-}Cl_{\mathbf{C}}\lambda\wedge\alpha\text{-}Cl_{\mathbf{C}}(\mathbf{C}\lambda)$. Since $\mathbf{C}$ satisfies the involutive condition $\mathbf{C}(\mathbf{C}\lambda)=\lambda$, that implies $\alpha\text{-}Bd_{\mathbf{C}}\lambda=\alpha\text{-}Cl_{\mathbf{C}}(\mathbf{C}\lambda)\wedge pCl_{\mathbf{C}}\mathbf{C}(\mathbf{C}\ )$. Again by using Definition 4.1, $\alpha\text{-}Bd_{\mathbf{C}}\lambda=\alpha\text{-}Bd_{\mathbf{C}}(\mathbf{C}\lambda)$.

The following example shows that, the word "involutive" can not be dropped from the hypothesis of Proposition 4.2.

Example 4.3

Let $X=\{a,b,c\}$ and $\tau=\{0,\{a_{.6},b_{.7},c_{.8}\},\{a_{.8},b_1,c_{.7}\},\{a_{.8},b_{.6},c_{.7}\},\{a_{.8},b_1,c_{.8}\},\{a_{.6},b_{.7},c_{.7}\},\{a_{.8},b_{.7},c_{.8}\},\{a_{.6},b_{.6},c_{.7}\},\{a_{.2},b_0,c_{.1}\}1\}$.

Let $\mathbf{C}(x)=\dfrac{1-x}{1+x^2}$, $0\leq x\leq 1$, be the

complement function. We note that the complement function $\mathbf{C}$ does not satisfy the involutive condition. The family of all fuzzy $\mathbf{C}$ -closed sets is $\mathbf{C}(\tau)=\{0,\{a_{.294}, b_{.201}, c_{.122}\}, \{a_{.122}, b_0, c_{.201}\}, \{a_{.122}, b_{.294}, c_{.201}\}, \{a_{.122}, b_0, c_{.122}\}, \{a_{.294}, b_{.201}, c_{.201}\}, \{a_{.122}, b_{.201}, c_{.122}\}, \{a_{.294}, b_{.294}, c_{.201}\}, \{a_{.769}, b_1, c_{.891}\}, 1\}$.
Let $\lambda = \{a_{.2}, b_0, c_{.1}\}$. Then it can be calculated that $\alpha\text{-}Cl_{\mathbf{C}}\lambda = \{a_{.2}, b_0, c_{.1}\}$.
Now $\mathbf{C}\ \lambda = \{a_{.769}, b_1, c_{.891}\}$ and the value of $\alpha\text{-}Cl_{\mathbf{C}}\mathbf{C}\ \lambda=\{a_{.769}, b_1, c_{.891}\}$. Hence $\alpha\text{-}Bd_{\mathbf{C}}\lambda = \alpha\text{-}Cl_{\mathbf{C}}\lambda\wedge\alpha\text{-}Cl_{\mathbf{C}}(\mathbf{C}\ \lambda) = \{a_{.2}, b_0, c_{.1}\}$. Also $\mathbf{C}(\mathbf{C}\ \lambda) = \{a_{.12}, b_0, c_{.0607}\}$, $\alpha\text{-}Cl_{\mathbf{C}}\mathbf{C}(\mathbf{C}\ \lambda) = \{a_{.12}, b_0, c_{.0607}\}$. $\alpha\text{-}Bd_{\mathbf{C}}\mathbf{C}\ \lambda = \alpha\text{-}Cl_{\mathbf{C}}\mathbf{C}\ \lambda \wedge \alpha\text{-}Cl_{\mathbf{C}}\mathbf{C}(\mathbf{C}\ \lambda) = \{a_{.12}, b_0, c_{.0607}\}$. This implies that $\alpha\text{-}Bd_{\mathbf{C}}\ \lambda \neq \alpha\text{-}Bd_{\mathbf{C}}\mathbf{C}\ \lambda$.

Proposition 4.4

Let (X,τ) be a fuzzy topological space and $\mathbf{C}$ be a complement function that satisfies the monotonic and involutive conditions. If λ is fuzzy $\mathbf{C}$ - α- closed, then $\alpha\text{-}Bd_{\mathbf{C}}\ \lambda \leq \lambda$.

Proof.

Let λ be fuzzy $\mathbf{C}$ - α-closed. By using Definition 4.1, $\alpha\text{-}Bd_{\mathbf{C}}\ \lambda = \alpha\text{-}Cl_{\mathbf{C}}\ \lambda\wedge\alpha\text{-}Cl_{\mathbf{C}}(\mathbf{C}\ \lambda)$. Since $\mathbf{C}$ satisfies the monotonic and involutive conditions, by using Proposition 3.6(ii), we have $\alpha\text{-}Cl_{\mathbf{C}}\ \lambda = \lambda$. Hence $\alpha\text{-}Bd_{\mathbf{C}}\ \lambda \leq \alpha\text{-}Cl_{\mathbf{C}}\lambda = \lambda$.

The following example shows that if the complement function $\mathbf{C}$ does not satisfy the monotonic and involutive conditions, then the conclusion of Proposition 4.4 is false.

Example 4.5

Let $X = \{a, b\}$ and $\tau = \{0, \{a_{.5}, b_{.6}\}, \{a_{.75}, b_{.2}\}, \{a_{.5}, b_{.2}\}, \{a_{.75}, b_{.6}\}, 1\}$.

Let $\mathbf{C}(x) = \dfrac{2x}{1+x}$, $0 \leq x \leq 1$, be a complement function. From this, we see that the complement function $\mathbf{C}$ does not satisfy the monotonic and involutive conditions. The family of all fuzzy $\mathbf{C}$ -closed sets is given by $\mathbf{C}(\tau) = \{0, \{a_{.667}, b_{.75}\}, \{a_{.857}, b_{.333}\}, \{a_{.667}, b_{.333}\}, \{a_{.857}, b_{.75}\}, 1\}$.
Let $\lambda = \{a_{.665}, b_{.462}\}$, it can be found that $Cl_{\mathbf{C}}\ \lambda = \{a_{.667}, b_{.33}\}$ and $Int\ Cl_{\mathbf{C}}\ \lambda = \{a_{.5}, b_{.2}\}$. That implies $Cl_{\mathbf{C}}\ Int\ Cl_{\mathbf{C}}\ \lambda = \{a_{.667}, b_{.33}\} \leq \lambda$. This shows that λ is fuzzy $\mathbf{C}$ - α-closed. Further it can be calculated that $\alpha\text{-}Cl_{\mathbf{C}}\lambda = \{a_{.667}, b_{.75}\}$. Now $\mathbf{C}\ \lambda = \{a_{.8}, b_{.857}\}$ and $\alpha\text{-}Cl_{\mathbf{C}}\mathbf{C}\ \lambda = \{1\}$. Hence $\alpha\text{-}Bd_{\mathbf{C}}\ \lambda = \alpha\text{-}Cl_{\mathbf{C}}\lambda\wedge\alpha\text{-}Cl_{\mathbf{C}}(\mathbf{C}\ \lambda) = \{a_{.667}, b_{.75}\}$. This implies that $\alpha\text{-}Bd_{\mathbf{C}}\ \lambda\ \ \lambda$. This shows that the conclusion of Proposition 4.4 is false.

Proposition 4.6

Let (X,τ) be a fuzzy topological space and $\mathbf{C}$ be a complement function that satisfies the monotonic and involutive conditions. If λ is fuzzy $\mathbf{C}$ -pre open then $pBd_{\mathbf{C}}\lambda \leq \mathbf{C}\ \lambda$.

Proof.

Let λ be fuzzy $\mathbf{C}$ -α- open. Since $\mathbf{C}$ satisfies the involutive condition, this implies that $\mathbf{C}(\mathbf{C}\ \lambda)$ is fuzzy $\mathbf{C}$ -α-open. By using Lemma 2.8, $\mathbf{C}\ \lambda$ is fuzzy $\mathbf{C}$ - α-closed. Since C satisfies the monotonic and the involutive conditions, by using Proposition 4.4, $\alpha\text{-}Bd_{\mathbf{C}}(\mathbf{C}\ \lambda)\ \mathbf{C}\ \lambda$. Also by using Proposition 4.2, we get $\alpha\text{-}Bd_{\mathbf{C}}(\lambda)\leq\mathbf{C}\ \lambda$. This completes the proof.

Example 4.7

Let $X =\{a, b, c\}$ and $\tau = \{0, \{a_{.3}, b_{.5}\}, \{a_{.5}, b_{.2}, c_{.15}\}, \{a_{.5}, b_{.5}, c_{.15}\}, \{a_{.3}, b_{.2}\}, 1\}$.

Let $\mathbf{C}(x)=\dfrac{1-x^2}{(1+x)^3}$, $0 \le x \le 1$, be the complement function. We note that the complement function $\mathbf{C}$ does not satisfy the involutive condition. The family of all fuzzy $\mathbf{C}$ -closed sets is $\mathbf{C}(\tau)=\{0,\{a_{.414}, b_{.222}, c_1\}, \{a_{.222}, b_{.556}, c_{.642}\}, \{a_{.222}, b_{.222}, c_{.156}\},\{a_{.414}, b_{.642}, c_1\}, 1\}$. Let $\lambda = \{a_{.4}, b_{.122}, c_{.57}\}$, the value of $\alpha\text{-}Cl_{\mathbf{C}}\lambda = \{a_{.414}, b_{.222}, c_{.174}\}$ and $\mathbf{C}\lambda = \{a_{.306}, b_{.701}, c_{.174}\}$, it follows that $\alpha\text{-Bd}_{\mathbf{C}}\lambda = \alpha\text{-}Cl_{\mathbf{C}}\lambda \wedge \alpha\text{-}Cl_{\mathbf{C}}(\mathbf{C}\lambda) = \{a_{.306}, b_{.222}, c_{.642}\}$. This shows that $\alpha\text{-Bd}_{\mathbf{C}}\lambda \mathbf{C}\lambda$. Therefore the conclusion of Proposition 4.6 is false.

Proposition 4.8 :

Let (X, τ) be a fuzzy topological space and $\mathbf{C}$ be a complement function that satisfies the monotonic and involutive conditions. If λ μ and μ is fuzzy $\mathbf{C}$ -α-closed then $\alpha\text{-Bd}_{\mathbf{C}}\lambda \le \mu$.

Proof.

Let $\lambda \le \mu$ and μ be fuzzy $\mathbf{C}$ -α-closed. Since $\mathbf{C}$ satisfies the monotonic and involutive conditions, by using Proposition 3.6(iv), we have $\lambda \le \mu$ implies $\alpha\text{-}Cl_{\mathbf{C}}\lambda \le \alpha\text{-}Cl_{\mathbf{C}}\mu$. By using Definition 4.1, $\alpha\text{-Bd}_{\mathbf{C}}\lambda = \alpha\text{-}Cl_{\mathbf{C}}\lambda \wedge \alpha\text{-}Cl_{\mathbf{C}}(\mathbf{C}\lambda)$. Since $\alpha\text{-}Cl_{\mathbf{C}}\lambda \le \alpha\text{-}Cl_{\mathbf{C}}\mu$, we have $\alpha\text{-Bd}_{\mathbf{C}}\lambda \le \alpha\text{-}Cl_{\mathbf{C}}\mu \wedge \alpha\text{-}Cl_{\mathbf{C}}(\mathbf{C}\lambda) \le \alpha\text{-}Cl_{\mathbf{C}}\mu$. Again by using Proposition 3.6 (ii), we have $\alpha\text{-}Cl_{\mathbf{C}}\mu = \mu$. This implies that $\alpha\text{-Bd}_{\mathbf{C}}\lambda \le \mu$.

The following example shows that if the complement function $\mathbf{C}$ does not satisfy the monotonic and involutive conditions, then the conclusion of Proposition 4.8 is false.

Example 4.9 :

Let $X = \{a, b\}$ and $\tau = \{0, \{a_{.6}, b_{.9}\}, \{a_{.7}, b_{.3}\}, \{a_{.6}, b_{.3}\}, \{a_{.7}, b_{.9}\}, 1\}$.

Let $\mathbf{C}(x) = \dfrac{2x}{1+x}$, $0 \le x \le 1$, be a complement function. From this, we see that the complement function $\mathbf{C}$ does not satisfy the monotonic and involutive conditions. The family of all fuzzy $\mathbf{C}$ -closed sets is given by $\mathbf{C}(\tau)=\{0, \{a_{.75}, b_{.947}\}, \{a_{.8235}, b_{.462}\}, \{a_{.75}, b_{.462}\},\{a_{.8235}, b_{.947}\}, 1\}$. Let $\lambda = \{a_{.7}, b_{.45}\}$ and $\mu = \{a_{.76}, b_{.5}\}$. Then it can be found that $Int\ Cl_{\mathbf{C}}\ \mu = \{a_{.7}, b_{.3}\}$ and $Cl_{\mathbf{C}}\ Int\ Cl_{\mathbf{C}}\ \mu = \{a_{.75}, b_{.462}\}$. That implies $Cl_{\mathbf{C}}\ Int\ Cl_{\mathbf{C}}\ \mu \le \mu$. This show that μ is fuzzy $\mathbf{C}$ -α-closed. It can be computed that $\alpha\text{-}Cl_{\mathbf{C}}\lambda = \{a_{.8}, b_{.47}\}$. Now $\mathbf{C}\lambda = \{a_{.824}, b_{.62}\}$ and $\alpha\text{-}Cl_{\mathbf{C}}\mathbf{C}\lambda = \{a_{.824}, b_{.47}\}$. $\alpha\text{-Bd}_{\mathbf{C}}\lambda = \alpha\text{-}Cl_{\mathbf{C}}\lambda \wedge \alpha\text{-}Cl_{\mathbf{C}}(\mathbf{C}\ ) = \{a_{.8}, b_{.47}\}$. This shows that $\alpha\text{-Bd}_{\mathbf{C}}\lambda\ \mu$. Therefore the conclusion of Proposition 4.8 is false.

Proposition 4.10

Let (X,τ) be a fuzzy topological space and $\mathbf{C}$ be a complement function that satisfies the monotonic and involutive conditions. If $\lambda \le \mu$ and μ is fuzzy $\mathbf{C}$ - α-open then $\alpha\text{-Bd}_{\mathbf{C}}\lambda \le \mathbf{C}\mu$.

Proof.

Let $\lambda \le \mu$ and μ is fuzzy $\mathbf{C}$ -α-open. Since $\mathbf{C}$ satisfies the monotonic condition, by using Proposition 3.6(iv), we have $\mathbf{C}\mu \le \mathbf{C}\lambda$ that implies $\alpha\text{-}Cl_{\mathbf{C}}\mathbf{C}\mu \le \alpha\text{-}Cl_{\mathbf{C}}\mathbf{C}\lambda$. By using Definition 4.1, $\alpha\text{-Bd}_{\mathbf{C}}\lambda = \alpha\text{-}Cl_{\mathbf{C}}\lambda \wedge \alpha\text{-}Cl_{\mathbf{C}}\mathbf{C}\lambda$. Taking complement on both sides, we get $\mathbf{C}(\alpha\text{-Bd}_{\mathbf{C}}\lambda)=\mathbf{C}(\alpha\text{-}Cl_{\mathbf{C}}\lambda \wedge \alpha\text{-}Cl_{\mathbf{C}}(\mathbf{C}\lambda))$. Since $\mathbf{C}$ satisfies the monotonic and involutive conditions, by using Lemma 2.1, we have $\mathbf{C}(\alpha\text{-Bd}_{\mathbf{C}}\lambda) = \mathbf{C}(\alpha\text{-}Cl_{\mathbf{C}}\lambda)\mathbf{C}(\alpha\text{-}Cl_{\mathbf{C}}(\mathbf{C}\lambda))$. Since $\alpha\text{-}Cl_{\mathbf{C}}\mathbf{C}\mu \le \alpha\text{-}Cl_{\mathbf{C}}\mathbf{C}\lambda$, $\mathbf{C}(\text{-Bd}_{\mathbf{C}}\lambda) \ge \mathbf{C}(\alpha\text{-}Cl_{\mathbf{C}}\mathbf{C}\mu) \vee \mathbf{C}(\alpha\text{-}Cl_{\mathbf{C}}\lambda)$, by using

Proposition 3.5(ii), $\mathbf{C}\,(\alpha\text{-}Bd_{\mathbf{C}}\lambda) \geq \alpha\text{-}Int_{\mathbf{C}}\,\mu \vee \alpha\text{-}Int_{\mathbf{C}}\mathbf{C}\,\lambda \geq \alpha\text{-}Int_{\mathbf{C}}\,\mu$. Since μ is fuzzy $\mathbf{C}$-α-open, $\mathbf{C}\,(\alpha\text{-}Bd_{\mathbf{C}}\lambda) \geq \mu$. Since $\mathbf{C}$ satisfies the monotonic conditions, $\alpha\text{-}Bd_{\mathbf{C}}\lambda\ \mathbf{C}\,\mu$.

The following example shows that if the complement function $\mathbf{C}$ does not satisfy the monotonic and involutive conditions, then the conclusion of Proposition 4.10 is false.

Example 4.1 :

Let X = {a, b} and τ = {0, {a.$_6$, b.$_9$}, {a.$_7$, b.$_3$}, {a.$_6$, b.$_3$}, {a.$_7$, b.$_9$}, 1}. Let $\mathbf{C}\,(x) = \dfrac{2x}{1+x}$, $0 \leq x \leq 1$, be a complement function. From this, we see that the complement function $\mathbf{C}$ does not satisfy the monotonic and involutive conditions. The family of all fuzzy $\mathbf{C}$-closed sets is given by $\mathbf{C}\,(\tau)$={0, {a.$_{75}$, b.$_{947}$}, {a.$_{8235}$, b.$_{462}$}, {a.$_{75}$, b.$_{462}$},{a.$_{8235}$, b.$_{947}$}, 1}. Let λ= {a.$_6$, b.$_3$} and μ = {a.$_{65}$, b.$_4$}. Then it can be evaluated that *Int* λ = {a.$_6$, b.$_3$} and *Int* $Cl_{\mathbf{C}}$ *Int* λ = {a.$_{75}$, b.$_{462}$}. Thus we see that $\lambda \leq Int\ Cl_{\mathbf{C}}(Int\lambda)$. By using Lemma 2.8, λ is fuzzy $\mathbf{C}$-α-open. It can be computed that $\alpha\text{-}Cl_{\mathbf{C}}\lambda$= {a.$_{85}$, b.$_{632}$}. Now $\mathbf{C}\,\lambda$ = {a.$_{75}$, b.$_{462}$} and $\alpha\text{-}Cl_{\mathbf{C}}C\ \lambda$ = {a.$_{85}$, b.$_{632}$}. $\alpha\text{-}Bd_{\mathbf{C}}\lambda = \alpha\text{-}Cl_{\mathbf{C}}\lambda \wedge \alpha\text{-}Cl_{\mathbf{C}}(\mathbf{C}\,\lambda)$ = {a.$_{85}$, b.$_{632}$}. This shows that $\alpha\text{-}Bd_{\mathbf{C}}\,\lambda \not\leq \mathbf{C}\,\mu$.

Proposition 4.12 :

Let (X,τ) be a fuzzy topological space. Let $\mathbf{C}$ be a complement function that satisfies the monotonic and involutive conditions. Then for any fuzzy subset λ of X, we have $\mathbf{C}\,(\alpha\text{-}Bd_{\mathbf{C}}\lambda) = \alpha\text{-}Int_{\mathbf{C}}\,\lambda \vee \alpha\text{-}Int_{\mathbf{C}}\,(\mathbf{C}\,\lambda)$.

Proof.

By using Definition 4.1, $\alpha\text{-}Bd_{\mathbf{C}}\lambda = \alpha\text{-}Cl_{\mathbf{C}}\,\lambda \wedge \alpha\text{-}Cl_{\mathbf{C}}(\mathbf{C}\,\lambda)$. Taking complement on both sdes, we get $\mathbf{C}\,(\alpha\text{-}Bd_{\mathbf{C}}\lambda) = \mathbf{C}\,(\alpha\text{-}Cl_{\mathbf{C}}\,\lambda \wedge \alpha\text{-}Cl_{\mathbf{C}}(\mathbf{C}\,))$. Since $\mathbf{C}$ satisfies the monotonic and involutive conditions, by using Lemma 2.4(ii), $\mathbf{C}\,(\alpha\text{-}Bd_{\mathbf{C}}\lambda) = \mathbf{C}\,(\alpha\text{-}Cl_{\mathbf{C}}\lambda) \vee \mathbf{C}\,(\alpha\text{-}Cl_{\mathbf{C}}\,(\mathbf{C}\,\lambda))$. Also by using Proposition 3.6(ii), that implies $\mathbf{C}\,(\alpha\text{-}Bd_{\mathbf{C}}\lambda) = \text{-}Int_{\mathbf{C}}(\mathbf{C}\,\lambda) \vee \alpha\text{-}Int_{\mathbf{C}}\,(\mathbf{C}\,(\mathbf{C}\,\lambda))$. Since $\mathbf{C}$ satisfies the involutive condition, $\mathbf{C}\,(\alpha\text{-}Bd_{\mathbf{C}}\lambda) = \alpha\text{-}Int_{\mathbf{C}}\,\lambda \vee \alpha\text{-}Int_{\mathbf{C}}\,(\mathbf{C}\,\lambda)$.

The following example shows that if the monotonic and involutive conditions of the complement function $\mathbf{C}$ can be dropped, then the conclusion of Proposition 4.12 is false.

Example 4.13:

Let X = {a, b} and τ = {0 , {a.$_3$, b.$_8$}, {a.$_2$, b.$_5$}, {a.$_7$, b.$_1$}, {a.$_3$, b.$_5$}, {a.$_3$, b.$_1$}, {a.$_2$, b.$_1$}, {a.$_7$, b.$_8$}, {a.$_7$, b.$_5$}, 1}. Let $\mathbf{C}\,(x) = \sqrt{x}$, $0 \leq x \leq 1$ be the complement function. From this example, we see that $\mathbf{C}$ does not satisfy the monotonic and involutive conditions. The family of all fuzzy $\mathbf{C}$-closed sets is $\mathbf{C}\,(\tau)$ = {0, {a.$_{548}$, b.$_{894}$}, {a.$_{447}$, b.$_{707}$},{a.$_{837}$, b.$_{316}$}, {.$_{548}$, b.$_{707}$},{a.$_{548}$, b.$_{316}$}, {a.$_{447}$, b.$_{316}$},{a.$_{837}$, b.$_{894}$}, {a.$_{837}$, b.$_{707}$},1}.

Let λ = {a.$_6$, b.$_3$}. Then it can be evaluated that $\alpha\text{-}Int_{\mathbf{C}}\,\lambda$ = {a.$_3$, b.$_1$}, $\mathbf{C}\,\lambda$ = {a.$_{775}$, b.$_{548}$} and $\alpha\text{-}Int_{\mathbf{C}}\,\mathbf{C}\,\lambda$ = {a.$_7$, b.$_5$}. Thus we see that $\alpha\text{-}Int_{\mathbf{C}}\,\lambda \vee \alpha\text{-}Int_{\mathbf{C}}\,\mathbf{C}\,\lambda$ = {a.$_{775}$, b.$_{548}$}. It can be computed that $\alpha\text{-}Cl_{\mathbf{C}}\lambda$ = {a.$_5$, b.$_8$}. Now $\mathbf{C}\,\lambda$ = {a.$_{775}$, b.$_{548}$}, $\alpha\text{-}Cl_{\mathbf{C}}\,\mathbf{C}\,\lambda$ = {a.$_{837}$, b.$_{707}$} and $\alpha\text{-}Bd_{\mathbf{C}}\,\lambda = \alpha\text{-}Cl_{\mathbf{C}}\lambda\ \ \alpha\text{-}Cl_{\mathbf{C}}\,(\mathbf{C}\,\lambda)$ = {a.$_5$, b.$_{707}$}. Also $\mathbf{C}\,(\alpha\text{-}Bd_{\mathbf{C}}\lambda)$ = {a.$_{707}$, b.$_{840}$}. Thus we see that $\mathbf{C}\,(\alpha\text{-}Bd_{\mathbf{C}}\,\lambda) \neq \alpha\text{-}Int_{\mathbf{C}}\,\lambda \vee \alpha\text{-}Int_{\mathbf{C}}\,\mathbf{C}\,\lambda$. Therefore the conclusion of Proposition 4.12 is false.

Proposition 4.14:

Let (X,τ) be a fuzzy topological space. Let $\mathbf{C}$ be a complement function that satisfies the monotonic and involutive conditions. Then for any fuzzy subset λ of X, we have $\alpha\text{-}Bd_{\mathbf{C}}(\lambda) = \alpha\text{-}Cl_{\mathbf{C}}(\lambda) \wedge \mathbf{C}(\alpha\text{-}Int_{\mathbf{C}}(\lambda))$.

Proof.

By using Definition 4.1, we have $\alpha\text{-}Bd_{\mathbf{C}}(\lambda) = \alpha\text{-}Cl_{\mathbf{C}}(\lambda) \wedge \alpha\text{-}Cl_{\mathbf{C}}(\mathbf{C}\,\lambda)$. Since $\mathbf{C}$ satisfies the monotonic and involutive conditions, by using Proposition 3.5(ii), we have $\alpha\text{-}Bd_{\mathbf{C}}(\lambda) = \alpha\text{-}Cl_{\mathbf{C}}(\lambda) \wedge \mathbf{C}(\alpha\text{-}Int_{\mathbf{C}}(\lambda))$.

The next example shows that if the complement function $\mathbf{C}$ does not satisfy the monotonic and involutive conditions, then the conclusion of Proposition 4.14 is false.

Example 4.15:

Let $X = \{a, b, c\}$ and $\tau = \{0, \{a_{.2}, b_{.6}, c_{.2}\}, \{a_{.7}, b_{.3}, c_{.7}\}, \{a_{.2}, b_{.3}, c_{.2}\}, \{a_{.7}, b_{.6}, c_{.7}\}, 1\}$. Let $\mathbf{C}(x) = \dfrac{1-x^3}{(1+x)^2}$, $0 \le x \le 1$, be the complement function. We note that the complement function $\mathbf{C}$ does not satisfy the involutive condition. The family of all fuzzy $\mathbf{C}$-closed sets is $\mathbf{C}(\tau)=\{0, \{a_{.689}, b_{.3062}, c_{.689}\}, \{a_{.227}, b_{.576}, c_{.227}\}, \{a_{.689}, b_{.576}, c_{.682}\}, \{a_{.227}, b_{.3062}, c_{.227}\}, 1\}$. Let $\lambda = \{a_{.5}, b_{.3062}, c_{.689}\}$, the value of $\alpha\text{-}Cl_{\mathbf{C}}\lambda = \{a_{.5}, b_{.3062}, c_{.689}\}$ and $\mathbf{C}\lambda = \{a_{.389}, b_{.569}, c_{.478}\}$, it follows that $\alpha\text{-}Bd_{\mathbf{C}}\lambda = \alpha\text{-}Cl_{\mathbf{C}}\lambda \wedge \alpha\text{-}Cl_{\mathbf{C}}(\mathbf{C}\,\lambda) = \{a_{.389}, b_{.3062}, c_{.4}\}$. Also $\mathbf{C}(\alpha\text{-}Int_{\mathbf{C}}\,\lambda)=\{a_{.689}, b_{.576}, c_{.689}\}$. It follows that $\alpha\text{-}Cl_{\mathbf{C}}\lambda \wedge \mathrm{C}(\alpha\text{-}Int_{\mathbf{C}}\,\lambda) = \{a_{.227}, b_{.3062}, c_{.227}\}$. This shows that $\alpha\text{-}Bd_{\mathbf{C}}\lambda \neq \alpha\text{-}Cl_{\mathbf{C}}\lambda \wedge \mathbf{C}(\alpha\text{-}Int_{\mathbf{C}}\,\lambda)$. Therefore the conclusion of Proposition 4.14 is false.

Proposition 4.16 :

Let (X,τ) be a fuzzy topological space. Let $\mathbf{C}$ be a complement function that satisfies the monotonic and involutive conditions. Then for any subset λ of X, $\alpha\text{-}Bd_{\mathbf{C}}(\alpha\text{-}Int_{\mathbf{C}}(\lambda)) \le \alpha\text{-}Bd_{\mathbf{C}}(\lambda)$.

Proof.

Since the complement function $\mathbf{C}$ satisfies the monotonic and involutive conditions, by using Proposition 4.14, we have $\alpha\text{-}Bd_{\mathbf{C}}(\alpha\text{-}Int_{\mathbf{C}}(\lambda)) = \alpha\text{-}Cl_{\mathbf{C}}(\alpha\text{-}Int_{\mathbf{C}}(\lambda)) \wedge \mathbf{C}(\alpha\text{-}Int_{\mathbf{C}}(\alpha\text{-}Int_{\mathbf{C}}(\lambda)))$. Since $\alpha\text{-}Int_{\mathbf{C}}(\lambda)$ is fuzzy $\mathbf{C}$-α-open, $\alpha\text{-}Bd_{\mathbf{C}}(\alpha\text{-}Int_{\mathbf{C}}(\lambda)) = \alpha\text{-}Cl_{\mathbf{C}}(\alpha\text{-}Int_{\mathbf{C}}(\lambda)) \wedge \mathbf{C}(\alpha\text{-}Int_{\mathbf{C}}(\lambda))$. Since $\alpha\text{-}Int_{\mathbf{C}}(\lambda) \le \lambda$, by using Proposition 3.6(ii), $\alpha\text{-}Cl_{\mathbf{C}}(\alpha\text{-}Int_{\mathbf{C}}(\lambda)) \le \alpha\text{-}Cl_{\mathbf{C}}(\lambda)$. Thus $\alpha\text{-}Bd_{\mathbf{C}}(\alpha\text{-}Int_{\mathbf{C}}(\lambda)) \le \alpha\text{-}Cl_{\mathbf{C}}(\lambda) \wedge \mathbf{C}(\alpha\text{-}Int_{\mathbf{C}}(\lambda))$. Since $\mathbf{C}$ satisfies the monotonic and involutive conditions, by using Proposition 3.5, $\alpha\text{-}Bd_{\mathbf{C}}(\alpha\text{-}Int_{\mathbf{C}}(\lambda)) \le \alpha\text{-}Cl_{\mathbf{C}}(\lambda) \wedge \alpha\text{-}Cl_{\mathbf{C}}(\mathbf{C}\,\lambda)$. By using Definition 4.1, we have $\alpha\text{-}Bd_{\mathbf{C}}(sInt_{\mathbf{C}}(\lambda)) \le \alpha\text{-}Bd_{\mathbf{C}}(\lambda)$.

Proposition 4.17 :

Let (X,τ) be a fuzzy topological space. Let $\mathbf{C}$ be a complement function that satisfies the monotonic and involutive conditions. Then $\alpha\text{-}Bd_{\mathbf{C}}(\alpha\text{-}Cl_{\mathbf{C}}(\lambda)) \le \text{-}Bd_{\mathbf{C}}(\lambda)$.

Proof.

Since $\mathbf{C}$ satisfies the monotonic and involutive conditions, by using Proposition 4.14, $\alpha\text{-}Bd_{\mathbf{C}}(\alpha\text{-}Cl_{\mathbf{C}}(\lambda)) = \alpha\text{-}Cl_{\mathbf{C}}(\alpha\text{-}Cl_{\mathbf{C}}(\lambda)) \wedge \mathbf{C}(\alpha\text{-}Int_{\mathbf{C}}(\alpha\text{-}Cl_{\mathbf{C}}(\lambda)))$. By using Proposition 3.6(iii), we have $\alpha\text{-}Cl_{\mathbf{C}}(\alpha\text{-}Cl_{\mathbf{C}}(\lambda)) = \alpha\text{-}Cl_{\mathbf{C}}(\lambda)$, that implies $\alpha\text{-}Bd_{\mathbf{C}}(\alpha\text{-}Cl_{\mathbf{C}}(\lambda)) = \alpha\text{-}Cl_{\mathbf{C}}(\lambda)\mathbf{C}(\alpha\text{-}Int_{\mathbf{C}}(\alpha\text{-}Cl_{\mathbf{C}}(\lambda)))$. Since $\lambda \le \alpha\text{-}Cl_{\mathbf{C}}(\lambda)$, that implies $\alpha\text{-}Int_{\mathbf{C}}(\lambda) \le \alpha\text{-}Int_{\mathbf{C}}(Cl_{\mathbf{C}}(\lambda))$. Therefore, $\alpha\text{-}Bd_{\mathbf{C}}(\alpha\text{-}Cl_{\mathbf{C}}(\lambda)) \le \text{-}Cl_{\mathbf{C}}(\lambda) \wedge \mathbf{C}(\alpha\text{-}Int_{\mathbf{C}}(\lambda))$. By using Proposition 3.5 (ii), and by using Definition 4.1, we get $\alpha\text{-}Bd_{\mathbf{C}}(\alpha\text{-}Cl_{\mathbf{C}}$

$(\lambda)) \leq \alpha\text{-}Bd_{\mathbf{C}}(\lambda)$.

Theorem 4.18 :

Let (X,τ) be a fuzzy topological space. Let $\mathbf{C}$ be a complement function that satisfies the monotonic and involutive conditions. Then $\alpha\text{-}Bd_{\mathbf{C}}\ (\lambda\vee\mu)\alpha\text{-}Bd_{\mathbf{C}}\,\lambda\vee\alpha\text{-}Bd_{\mathbf{C}}\,\mu$.

Proof.

By using Definition 4.1, $\alpha\text{-}Bd_{\mathbf{C}}\ (\lambda\vee\mu) = \alpha\text{-}Cl_{\mathbf{C}}(\lambda\vee\mu)\wedge\alpha\text{-}Cl_{\mathbf{C}}(\mathbf{C}\,(\lambda\vee\mu))$. Since $\mathbf{C}$ satisfies the monotonic and involutive conditions, by using Proposition 3.7(i), that implies $\alpha\text{-}Bd_{\mathbf{C}}\ (\lambda\vee\mu) = (\alpha\text{-}Cl_{\mathbf{C}}(\lambda)\vee\alpha\text{-}Cl_{\mathbf{C}}\ (\mu))\wedge\alpha\text{-}Cl_{\mathbf{C}}\ (\mathbf{C}\ (\lambda\vee\mu))$. By using Lemma 2.4 and Proposition 3.7(ii), $\alpha\text{-}Bd_{\mathbf{C}}\ (\lambda\vee\mu) \leq (\alpha\text{-}Cl_{\mathbf{C}}\ (\lambda) \vee \alpha\text{-}Cl_{\mathbf{C}}\ (\mu)) \wedge (\alpha\text{-}Cl_{\mathbf{C}}(\mathbf{C}\ \lambda) \wedge \alpha\text{-}Cl_{\mathbf{C}}(\mathbf{C}\ \mu))$. That is, $\alpha\text{-}Bd_{\mathbf{C}}\ (\lambda\vee\mu) \leq (\alpha\text{-}Cl_{\mathbf{C}}(\lambda) \wedge \alpha\text{-}Cl_{\mathbf{C}}(\mathbf{C}\ ))\vee(\alpha\text{-}Cl_{\mathbf{C}}\ (\mu))\wedge\text{-}Cl_{\mathbf{C}}(\mathbf{C}\ \ \mu))$. Again by using Definition 4.1, $\alpha\text{-}Bd_{\mathbf{C}}\ (\lambda\vee\mu)\ \ \alpha\text{-}Bd_{\mathbf{C}}\ (\lambda) \vee \alpha\text{-}Bd_{\mathbf{C}}\ (\mu)$.

Theorem 4.19 :

Let (X,τ) be a fuzzy topological space. Suppose the complement function $\mathbf{C}$ satisfies the monotonic and involutive conditions. Then for any two fuzzy subsets λ and μ of a fuzzy topological space X, we have $\alpha\text{-}Bd_{\mathbf{C}}\ (\lambda\wedge\mu\ ) \leq\ (\alpha\text{-}Bd_{\mathbf{C}}\ (\lambda)\wedge\alpha\text{-}Cl_{\mathbf{C}}\ (\mu))\ \ (\alpha\text{-}Bd_{\mathbf{C}}\ (\mu)\wedge\alpha\text{-}Cl_{\mathbf{C}}\ (\lambda))$.

Proof.

By using Definition 4.1, we have $\alpha\text{-}Bd_{\mathbf{C}}(\lambda\wedge\mu) = \alpha\text{-}Cl_{\mathbf{C}}(\lambda\wedge\mu) \wedge \alpha\text{-}Cl_{\mathbf{C}}(\mathbf{C}\ (\lambda\wedge\mu))$. Since $\mathbf{C}$ satisfies the monotonic and involutive conditions, by using Proposition 3.7(i), Proposition 3.7 (ii) and by using Lemma 2.4(iv), we get $\alpha\text{-}Bd_{\mathbf{C}}(\lambda\wedge\mu) \leq (\alpha\text{-}Cl_{\mathbf{C}}(\lambda)\wedge\ \alpha\text{-}Cl_{\mathbf{C}}\ (\mu))\wedge\ (\alpha\text{-}Cl_{\mathbf{C}}(\mathbf{C}\ \lambda)\vee\alpha\text{-}Cl_{\mathbf{C}}(\mathbf{C}\ \mu))$ is equal to $[\alpha\text{-}Cl_{\mathbf{C}}(\lambda)\wedge\alpha\text{-}Cl_{\mathbf{C}}\ (\mathbf{C}\ \ \lambda)]\wedge(\alpha\text{-}Cl_{\mathbf{C}}(\mu))[\alpha\text{-}Cl_{\mathbf{C}}\ (\mu)\wedge\alpha\text{-}Cl_{\mathbf{C}}\ (\mathbf{C}\ \ \mu)]\ \wedge pCl_{\mathbf{C}}(\lambda)$. Again by Definition 4.1, we get $\alpha\text{-}Bd_{\mathbf{C}}(\lambda\wedge\mu\ ) \leq (\alpha\text{-}Bd_{\mathbf{C}}\ (\lambda)\wedge\alpha\text{-}Cl_{\mathbf{C}}\ (\mu))\vee(\alpha\text{-}Bd_{\mathbf{C}}(\mu)\wedge\alpha\text{-}Cl_{\mathbf{C}}\ (\lambda))$.

Proposition 4.20 :

Let (X, τ) be a fuzzy topological space. Suppose the complement function $\mathbf{C}$ satisfies the monotonic and involutive conditions. Then for any fuzzy subset λ of a fuzzy topological space X, we have (i) $\alpha\text{-}Bd_{\mathbf{C}}\ (\alpha\text{-}Bd_{\mathbf{C}}\ (\lambda)) \leq \alpha\text{-}Bd_{\mathbf{C}}\,(\lambda)$

(ii) $\alpha\text{-}Bd_{\mathbf{C}}\ \alpha\text{-}Bd_{\mathbf{C}}\ \alpha\text{-}Bd_{\mathbf{C}}\ \lambda \leq \alpha\text{-}Bd_{\mathbf{C}}\ \alpha\text{-}Bd_{\mathbf{C}}\,\lambda$.

Proof.

By using Definition 4.1, $\alpha\text{-}Bd_{\mathbf{C}}\lambda = \alpha\text{-}Cl_{\mathbf{C}}\,\lambda\wedge\alpha\text{-}Cl_{\mathbf{C}}(\mathbf{C}\ \lambda)$. We have $\alpha\text{-}Bd_{\mathbf{C}}\,\alpha\text{-}Bd_{\mathbf{C}}\,\lambda = \alpha\text{-}Cl_{\mathbf{C}}\ (\alpha\text{-}Bd_{\mathbf{C}}\,\lambda) \wedge \alpha\text{-}Cl_{\mathbf{C}}\ [\mathbf{C}\ (\alpha\text{-}Bd_{\mathbf{C}}\,\lambda)] \leq \alpha\text{-}Cl_{\mathbf{C}}(\alpha\text{-}Bd_{\mathbf{C}}\lambda)$. Since $\mathbf{C}$ satisfies the monotonic and involutive conditions, by using Proposition 3.6(ii), $\alpha\text{-}Cl_{\mathbf{C}}\,\lambda=\lambda$, where λ is fuzzy $\mathbf{C}$-α-closed. Here $\alpha\text{-}Bd_{\mathbf{C}}\ \lambda$ is fuzzy $\mathbf{C}$ -α-closed. So, $\alpha\text{-}Cl_{\mathbf{C}}(\alpha\text{-}Bd_{\mathbf{C}}\,\lambda) = \alpha\text{-}Bd_{\mathbf{C}}\,\lambda$. This implies that $\alpha\text{-}Bd_{\mathbf{C}}\ \alpha\text{-}Bd_{\mathbf{C}}\ \lambda \leq \alpha\text{-}Bd_{\mathbf{C}}\,\lambda$. This proves (i). (ii) Follows from (i).

Proposition 4.21 :

Let λ be a fuzzy $\mathbf{C}$ -α-closed subset of a fuzzy topological space X and μ be a fuzzy $\mathbf{C}$ -α-closed subset of a fuzzy topological space Y, then $\lambda\times\mu$ is a fuzzy $\mathbf{C}$ -α- closed set of the fuzzy product space $X \times Y$ where the complement function $\mathbf{C}$ satisfies the monotonic and involutive conditions.

Proof.

Let λ be a fuzzy $\mathbf{C}$ -α-closed subset of a fuzzy topological space X. Then by applying Lemma 2.8, $\mathbf{C}\ \lambda$ is fuzzy $\mathbf{C}$ -α-open set in X. Also if $\mathbf{C}\ \lambda$ is fuzzy $\mathbf{C}$ -α-open set in X, then $\mathbf{C}\ \lambda \times 1$ is fuzzy $\mathbf{C}$ -α-open in $X \times Y$. Similarly let μ be a fuzzy $\mathbf{C}$ -α-closed subset of a fuzzy topological space X. Then by using Lemma 2.8, $\mathbf{C}\,\mu$ is fuzzy $\mathbf{C}$ -α-open set in Y. Also if $\mathbf{C}\,\mu$ is fuzzy $\mathbf{C}$ -α-open set in Y then $\mathbf{C}\,\mu \times 1$ is fuzzy $\mathbf{C}$ -α-open in $X \times Y$. Since the arbitrary union of fuzzy $\mathbf{C}$ -α-open sets is fuzzy$\mathbf{C}$ -α-open. So, $\mathbf{C}\,\lambda \times 1 \vee 1 \times \mathbf{C}\,\mu$ is fuzzy $\mathbf{C}$ -α-open in $X \times Y$. By using Lemma 2.13, $\mathbf{C}\,(\lambda \times \mu) = \mathbf{C}\,\lambda \times 1 \vee 1 \times \mathbf{C}\,\mu$, hence $\mathbf{C}\,(\lambda \times \mu)$ is fuzzy $\mathbf{C}$ -α-open. By using Lemma 2.8, $\lambda \times \mu$ is fuzzy$\mathbf{C}$ - α- closed of the fuzzy product space $X \times Y$.

Theorem 4.22 :

Let $\mathbf{C}$ be a complement function that satisfies the monotonic and involutive conditions. If λ is a fuzzy subset of a fuzzy topological space X and μ is a fuzzy subset of a fuzzy topological space Y, then

(i) $\alpha\text{-}Cl_{\mathbf{C}}\,\lambda \times \alpha\text{-}Cl_{C}\,\mu \geq \alpha\text{-}Cl_{\mathbf{C}}\,(\lambda \times \mu)$

(ii) $\alpha\text{-}Int_{\mathbf{C}}\,\lambda \times \alpha\text{-}Int_{\mathbf{C}}\,\mu \leq \text{-}Int_{\mathbf{C}}\,(\lambda \times \mu)$.

Proof.

By using Definition 2.20, $(\alpha\text{-}Cl_{\mathbf{C}}\,\lambda \times \alpha\text{-}Cl_{\mathbf{C}}\,\mu)\,(x, y) = \min\{\alpha\text{-}Cl_{\mathbf{C}}\,\lambda(x), \alpha\text{-}Cl_{\mathbf{C}}\,\mu(y)\} \geq \min\{\lambda(x), \mu(y)\} = (\lambda \times \mu)\,(x, y)$. This shows that $\alpha\text{-}Cl_{\mathbf{C}}\,\lambda \;\; \alpha\text{-}Cl_{\mathbf{C}}\,\mu \geq (\lambda \times \;)$. By using Proposition 3.6, $\alpha\text{-}Cl_{\mathbf{C}}\,(\lambda \times \;) \leq \alpha\text{-}Cl_{\mathbf{C}}\,(\alpha\text{-}Cl_{\mathbf{C}}\,\lambda \times \text{-}Cl_{\mathbf{C}}\,\mu) = \alpha\text{-}Cl_{\mathbf{C}}\,\lambda \times \alpha\text{-}Cl_{\mathbf{C}}\,\mu$. By using Definition 2.10, $(\alpha\text{-}Int_{\mathbf{C}}\,\lambda \times \alpha\text{-}Int_{\mathbf{C}}\,\mu)\,(x, y) = \min\{\alpha\text{-}Int_{\mathbf{C}}\,\lambda(x), \alpha\text{-}Int_{\mathbf{C}}\,\mu(y)\} \leq \min\{\lambda(x), \mu(y)\} = (\lambda \times \mu)(x, y)$. This shows that $\alpha\text{-}Int_{\mathbf{C}}\,\lambda \times \alpha\text{-}Int_{\mathbf{C}}\,\mu \leq (\lambda \times \mu)$. By using Proposition 3.2, $\alpha\text{-}Int_{\mathbf{C}}\,(\alpha\text{-}Int_{\mathbf{C}}\,\lambda \times \alpha\text{-}Int_{\mathbf{C}}\,\mu) \leq \alpha\text{-}Int_{\mathbf{C}}\,(\lambda \times \mu)$, that implies $\alpha\text{-}Int_{\mathbf{C}}\,\lambda \times \alpha\text{-}Int_{\mathbf{C}}\,\mu \leq \alpha\text{-}Int_{\mathbf{C}}\,(\lambda \times \mu)$.

Theorem 4.23 :

Let X and Y be $\mathbf{C}$ -product related fuzzy topological spaces. Then for a fuzzy subset λ of X and a fuzzy subset μ of Y,

(i) $\alpha\text{-}Cl_{\mathbf{C}}\,(\lambda \times \mu) = \alpha\text{-}Cl_{\mathbf{C}}\,\lambda \;\; \alpha\text{-}Cl_{\mathbf{C}}\,\mu$

(ii) $\alpha\text{-}Int_{\mathbf{C}}\,(\lambda \times \mu) = \alpha\text{-}Int_{\mathbf{C}}\,\lambda \times \alpha\text{-}Int_{\mathbf{C}}\,\mu$.

Proof.

By using Theorem 4.22, it is sufficient to show that $\alpha\text{-}Cl_{\mathbf{C}}\,(\lambda \times \mu) \geq \alpha\text{-}Cl_{\mathbf{C}}\,\lambda \times \alpha\text{-}Cl_{\mathbf{C}}\,\mu$. By using Definition 3.4, we have $\alpha\text{-}Cl_{\mathbf{C}}\,(\lambda \times \mu) = \inf\{\mathbf{C}\,(\lambda_\alpha \times \mu_\beta) : \mathbf{C}\,(\lambda_\alpha \times \mu_\beta) \geq \lambda \;\; \mu$ where λ_α and μ_β are fuzzy $\mathbf{C}$ -α-open$\}$. By using Lemma 2.12, we have

$\alpha\text{-}Cl_{\mathbf{C}}\,(\lambda \times \;) = \inf\{\mathbf{C}\,\lambda_\alpha \times 1 \vee 1 \times \mathbf{C}\,\lambda_\beta : \mathbf{C}\,\lambda_\alpha \times 1 \vee 1\mathbf{C}\,\mu_\alpha \geq \; \times \mu\}$

$= \inf\{\mathbf{C}\,\lambda_\alpha \times 1 \vee 1 \times \mathbf{C}\,\mu_\beta : \mathbf{C}\,\lambda_\beta \geq \lambda$ or $\mathbf{C}\,\mu_\beta \geq \;\}$

$= \min(\inf\{\mathbf{C}\,\lambda_\alpha \times 1 \vee 1 \times \mathbf{C}\,\mu_\beta : C\,\lambda_\alpha \geq \lambda\}, \inf\{\mathbf{C}\,\lambda_\alpha \times 1 \vee 1 \times \mathbf{C}\,\mu_\beta : \mathbf{C}\,\mu_\beta \geq \;\})$.

Now $\inf\{\mathbf{C}\,\lambda_\alpha \times 1 \vee 1\;\mathbf{C}\,\mu_\beta : \mathbf{C}\,\lambda_\alpha \geq \lambda\} \geq \inf\{\mathbf{C}\,\lambda_\alpha \times 1 : \mathbf{C}\,\lambda_\alpha \geq \lambda\}$

$= \inf\{\mathbf{C}\,\lambda_\alpha : \mathbf{C}\,\lambda_\alpha \geq \lambda\} \times 1$

$= (\alpha\text{-}Cl_{\mathbf{C}}\,\lambda) \times 1$.

Also $\inf\{\mathbf{C}\,_\alpha \times 1 \vee 1 \times \mathbf{C}\,\mu_\beta : \mathbf{C}\,\mu_\beta \geq \mu\} \geq \inf\{1 \times \mathbf{C}\,\mu_\beta : \mathbf{C}\,\mu_\beta \geq \mu\}$

$= 1 \times \inf\{\mathbf{C}\,\mu_\beta : \mathbf{C}\,\mu_\beta \geq \;\}$

$= 1 \times \alpha\text{-}Cl_{\mathbf{C}}\,\mu$.

The above discussions imply that $\alpha\text{-}Cl_{\mathbf{C}}\,(\lambda \times \mu)$ $\min(\alpha\text{-}Cl_{\mathbf{C}}\,\lambda \times 1, 1 \times \alpha\text{-}Cl_{\mathbf{C}}\,\mu)$

$= \alpha\text{-}Cl_{\mathbf{C}}\,\lambda \times \alpha\text{-}Cl_{\mathbf{C}}\,\mu$.

(ii) follows from (i) and using Proposition 3.5.

Theorem 4.24 :

Let X_i, $i = 1, 2 \ldots.n$, be a family of $\mathbf{C}$ -product related fuzzy topological spaces. If λ_i is a fuzzy subset of X_i, and the complement function$\mathbf{C}$ satisfies the monotonic and involutive conditions, then $\alpha\text{-Bd}_{\mathbf{C}}\,(\prod_{i=1}^{n} \lambda_i) = [\alpha\text{-Bd}_{\mathbf{C}}\,\lambda_1 \times \alpha\text{-}Cl_{\mathbf{C}}\,\lambda_2 \times \ldots \times \alpha\text{-}Cl_{\mathbf{C}}\,\lambda_n] \vee [\alpha\text{-}Cl_{\mathbf{C}}\,\lambda_1 \times \alpha\text{-Bd}_{\mathbf{C}}\,\lambda_2 \times \ldots \times \alpha\text{-}Cl_{\mathbf{C}}\,\lambda_n] \;\vee\; \ldots \vee [\alpha\text{-}Cl_{\mathbf{C}}\,\lambda_1 \times \alpha\text{-}Cl_{\mathbf{C}}$

$\lambda_2 \times \ldots\ldots \times \alpha\text{-Bd}_{\mathbf{C}}\lambda_n]$.

Proof.

It suffices to prove this for n = 2. By using Proposition 4.14, we have $\text{pBd}_{\mathbf{C}}(\lambda_1 \times \lambda_2) = \alpha\text{-}Cl_{\mathbf{C}}\,(\lambda_1 \times \lambda_2) \wedge \mathbf{C}\,(\text{-}Int_{\mathbf{C}}(\lambda_1 \times \lambda_2))$

$=(\alpha\text{-}Cl_{\mathbf{C}}\ \lambda_1 \times \alpha\text{-}Cl_{\mathbf{C}}\lambda_2)\wedge\mathbf{C}\,(\alpha\text{-}Int_{\mathbf{C}}\lambda_1 \times \alpha\text{-}Int_{\mathbf{C}}\lambda_2)$ [by using Theorem 4.23]

$=(\alpha\text{-}Cl_{\mathbf{C}}\ \lambda_1\times\alpha\text{-}Cl_{\mathbf{C}}\ \lambda_2)\wedge\mathbf{C}\ [(\alpha\text{-}Int_{\mathbf{C}}\ \lambda_1 \wedge\alpha\text{-}Cl_{\mathbf{C}}\ \lambda_1)\times(\alpha\text{-}Int_{\mathbf{C}}\lambda_2\wedge\alpha\text{-}Cl_{\mathbf{C}}\ \lambda_2)]$

$= (\alpha\text{-}Cl_{\mathbf{C}}\lambda_1 \times \alpha\text{-}Cl_{\mathbf{C}}\lambda_2) \wedge [\mathbf{C}\ (\alpha\text{-}Int_{\mathbf{C}}\lambda_1 \wedge \alpha\text{-}Cl_{\mathbf{C}}\lambda_1) \times 11 \times\mathbf{C}\ (\alpha\text{-}Int_{\mathbf{C}}\lambda_2 \wedge \alpha\text{-}Cl_{\mathbf{C}}\lambda_1)]$. [by Lemma 2.22]. Since $\mathbf{C}$ satisfies the monotonic and involutive conditions, by using Proposition 3.5(i), Proposition 3.5(i) and also by using Lemma 2.10, $\alpha\text{-Bd}_{\mathbf{C}}(\lambda_1 \times \lambda_2) = (\alpha\text{-}Cl_{\mathbf{C}}\ \lambda_1 \times \alpha\text{-}Cl_{\mathbf{C}}\ \lambda_2)\wedge[(\alpha\text{-}Cl_{\mathbf{C}}\mathbf{C}\ \lambda_1 \vee \alpha\text{-}Int_{\mathbf{C}}\mathbf{C}\ \lambda_1) \times 1\vee1 \times (\alpha\text{-}Cl_{\mathbf{C}}\ \mathbf{C}\ \lambda_2 \vee \alpha\text{-}Int_{\mathbf{C}}\ \mathbf{C}\ \lambda_2)]$.

$= (\alpha\text{-}Cl_{\mathbf{C}}\ \lambda_1 \times \alpha\text{-}Cl_{\mathbf{C}}\lambda_2) \wedge [\alpha\text{-}Cl_{\mathbf{C}}\ \mathbf{C}\ \lambda_1 \times 1\vee1 \times\alpha\text{-}Cl_{\mathbf{C}}\ \mathbf{C}\ \lambda_2]$

$= [(\alpha\text{-}Cl_{\mathbf{C}}\lambda_1\times\alpha\text{-}Cl_{\mathbf{C}}\lambda_2)\wedge(\alpha\text{-}Cl_{\mathbf{C}}\ (\mathbf{C}\ \lambda_1)\times1)] \vee [(\alpha\text{-}Cl_{\mathbf{C}}\ (\lambda_1) \times \alpha\text{-}Cl_{\mathbf{C}}\lambda_2) \wedge (1 \times \alpha\text{-}Cl_{\mathbf{C}}\,(\mathbf{C}\ \lambda_2))]$.

Again by using Lemma 2.18, we get $\alpha\text{-Bd}_{\mathbf{C}}(\lambda_1 \times \lambda_2)$

$= [(\alpha\text{-}Cl_{\mathbf{C}}\lambda_1\wedge\alpha\text{-}Cl_{\mathbf{C}}\ (\mathbf{C}\ \lambda_1)) \times (1\wedge\alpha\text{-}Cl_{\mathbf{C}}\ \lambda_2)] \vee [(\text{-}Cl_{\mathbf{C}}\ \lambda_2\wedge\text{-}Cl_{\mathbf{C}}\,(\mathbf{C}\ \lambda_2)) \times(1\wedge\alpha\text{-}Cl_{\mathbf{C}}\ \lambda_1)]$

$= [(\alpha\text{-}Cl_{\mathbf{C}}(\lambda_1) \wedge\alpha\text{-}Cl_{\mathbf{C}}\ (\mathbf{C}\ \lambda_1)) \times \alpha\text{-}Cl_{\mathbf{C}}\ (\lambda_2)\,] \vee [(\alpha\text{-}Cl_{\mathbf{C}}\ (\lambda_2)\wedge\alpha\text{-}Cl_{\mathbf{C}}(\mathbf{C}\ \lambda_2)) \times \alpha\text{-}Cl_{\mathbf{C}}\ (\lambda_1)]$

$\alpha\text{-Bd}_{\mathbf{C}}(\lambda_1 \times \lambda_2) = [\alpha\text{-Bd}_{\mathbf{C}}\ (\lambda_1) \times \alpha\text{-}Cl_{\mathbf{C}}\ (\lambda_2)] \vee [\alpha\text{-}Cl_{\mathbf{C}}\ (\lambda_1) \times \alpha\text{-Bd}_{\mathbf{C}}\ (\lambda_2)]$.

Theorem 4.25

Let f: X→Y be a fuzzy continuous function. Suppose the complement function $\mathbf{C}$ satisfies the monotonic and involutive conditions. Then

$\alpha\text{-Bd}_{\mathbf{C}}\ (f^{-1}(\mu)) \leq f^{-1}\ (\alpha\text{-Bd}_{\mathbf{C}}\ (\mu))$, for any fuzzy subset μ in Y.

Proof.

Let f be a fuzzy continuous function and μ be a fuzzy subset in Y. By using Definition 4.1, we have $\alpha\text{-Bd}_{\mathbf{C}}\ (f^{-1}(\mu)) = \alpha\text{-}Cl_{\mathbf{C}}\ (f^{-1}(\mu)) \wedge\alpha\text{-}Cl_{\mathbf{C}}\ (\mathbf{C}\ (f^{-1}(\mu))$. By using Lemma 2.19, $\alpha\text{-Bd}_{\mathbf{C}}\ (f^{-1}(\mu)) = \alpha\text{-}Cl_{\mathbf{C}}\ (f^{-1}(\mu))\wedge\alpha\text{-}Cl_{\mathbf{C}}\ (f^{-1}(\mathbf{C}\ \mu))$. Since f is fuzzy continuous and $f^{-1}(\mu) \leq f^{-1}(\alpha\text{-}Cl_{\mathbf{C}}\ (\mu))$, it follows that $\alpha\text{-}Cl_{\mathbf{C}}\ (f^{-1}(\mu)) \leq f^{-1}(\alpha\text{-}Cl_{\mathbf{C}}\ (\mu))$. This together with the above imply that $\alpha\text{-Bd}_{\mathbf{C}}\ (f^{-1}(\mu)) \leq f^{-1}(\alpha\text{-}Cl_{\mathbf{C}}\ (\mu))\wedge f^{-1}(\alpha\text{-}Cl_{\mathbf{C}}\ (\mathbf{C}\ \mu))$. By using Lemma 2.21, $\alpha\text{-Bd}_{\mathbf{C}}\ (f^{-1}(\mu)) \leq f^{-1}\ (\alpha\text{-}Cl_{\mathbf{C}}\ (\mu)\wedge\alpha\text{-}Cl_{\mathbf{C}}(\mathbf{C}\ \mu))$. That is $\alpha\text{-Bd}_{\mathbf{C}}\ (f^{-1}\ (\mu)) \leq f^{-1}(\alpha\text{-Bd}_{\mathbf{C}}\ (\mu))$.

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