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Decomposition of curvature tensor field in a Tachibana recurrent space

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Abstract

Takano[6] have studied decomposition of curvature tensor in a recurrent space. Sinha and Singh[5] have been studied on decomposition of recurrent curvature tensor fields in a Finsler space. Further, Negi and Rawat[4] have studied decomposition of recurrent curvature tensor field in a Kaehlerian space.

In the present paper, we have studied the decomposition of curvature tensor field R^h_{ijk} in terms of two vectors and a tensor field. Also several theorems have been investigated and proved.

Key words : Curvature tensor, Tachibana space, Recurrent space.

1. Introduction

An almost Tachibana space is an almost Hermite space $(F_i^h,\ g_{ij})$, where F_i^h is an almost complex structure and g_{ij} is the Hermite metric such that

$$F^h_{i,j} + F^h_{j,i} = 0, \tag{1.1}$$

where the comma (,) followed by an index denotes the operator of covariant differentiation with respect to the Riemannian metric tensor g_{ij}.

In an almost Tachibana space[1], we have

$$N^h_{j,i} = -4\left(F^a_{i,j}\right)F^h_a \tag{1.2}$$

where $F^h_{i,j}$ is pure in i and j and N^h_j is the Nijenhuis tensor[2]. When the Nijenhuis tensor vanishes, the almost Tachibana space is called a Tachibana space and denoted by T^c_n.

The Riemannian curvature tensor field R^h_{ijk} is defined as

$$R^h_{ijk} = \partial_i \begin{Bmatrix} h \\ j \quad k \end{Bmatrix} - \partial_j \begin{Bmatrix} h \\ i \quad k \end{Bmatrix} + \begin{Bmatrix} h \\ i \quad a \end{Bmatrix} \begin{Bmatrix} a \\ j \quad k \end{Bmatrix} - \begin{Bmatrix} h \\ j \quad a \end{Bmatrix} \begin{Bmatrix} a \\ i \quad k \end{Bmatrix}, \tag{1.3}$$

where $\partial_i = \frac{\partial}{\partial x^i}$ and $\{x^i\}$ denotes the real local coordinates.

The Ricci tensor and the Scalar curvature are given by

$R_{ij} = R^a_{aij}$ and $R = g^{ij} R_{ij}$ respectively. It is well known that these tensors satisfies the following identities

$$R^a_{ijk,a} = R_{jk,i} - R_{ik,j} \tag{1.4}$$

$$R_{,i} = 2R^a_{i,a} \tag{1.5}$$

$$F^a_i R_{aj} = -R_{ia} F^a_j , \tag{1.6}$$

and

$$F^a_i R^j_a = R^a_i F^j_a . \tag{1.7}$$

The Bianchi identities in T^c_n are given by

$$R^h_{ijk} + R^h_{jki} + R^h_{kij} = 0, \tag{1.8}$$

and

$$R^h_{ijk,a} + R^h_{ika,j} + R^h_{iaj,k} = 0. \tag{1.9}$$

The commutative formulae for the curvature tensor fields are given as follows

$$T^i_{,jk} - T^i_{,kj} = T^a R^i_{ajk} , \tag{1.10}$$

and

$$T^h_{i,ml} - T^h_{i,lm} = T^a_i R^h_{aml} - T^h_a R^a_{iml} . \tag{1.11}$$

A Tachibana space T^c_n is said to be Tachibana recurrent space[3], if the curvature tensor field satisfies the condition

$$R^h_{ijk,a} = \lambda_a R^h_{ijk} , \tag{1.12}$$

where λ_a is anon-zero vector and is known as recurrence vector field.

The space is said to be Ricci-recurrent space, if it satisfies the condition

$$R_{ij,a} = \lambda_a R_{ij} , \tag{1.13}$$

Multiplying the above equation (1.13) by g^{ij}, we have

$$R_{,a} = \lambda_a R. \tag{1.14}$$

2. Decomposition of Curvature Tensor Field $\boldsymbol{R^h_{ijk}}$ *:*

We consider the decomposition of curvature tensor field R^h_{ijk} in the following form

$$R^h_{ijk} = P^{,h} X_{,i} Y_{j,k} \tag{2.1}$$

where two vectors $P^{,h}$, $X_{,i}$ and a tensor field $Y_{j,k}$ are such that

$$\lambda_h P^{,h} = 1 . \tag{2.2}$$

Now, we have the following:

Theorem (2.1): Under the decomposition (2.1), the Bianchi identities for R^h_{ijk} takes the forms,

$$X_{,i} Y_{j,k} + X_{,j} Y_{k,i} + X_{,k} Y_{i,j} = 0. \tag{2.3}$$

and

$$\lambda_a Y_{j,k} + \lambda_j Y_{k,a} + \lambda_k Y_{a,j} = 0. \tag{2.4}$$

Proof. From (1.8) and (2.1), we get

$$P^{,h} (X_{,i} Y_{j,k} + X_{,j} Y_{k,i} + X_{,k} Y_{i,j}) = 0. \tag{2.5}$$

Multiplying (2.5) by λ_h and using (2.2), we get the required result (2.3).

Again, using (1.9), (1.12) and (2.1), we have

$$P^{,h} X_{,i}\left(\lambda_a Y_{j,k} + \lambda_j Y_{k,a} + \lambda_k Y_{a,j} = 0\right) \quad (2.6)$$

Multiplying (2.6) by λ_h and using (2.2), we get

$$\lambda_a Y_{j,k} + \lambda_j Y_{k,a} + \lambda_k Y_{a,j} = 0. \quad (\text{since } X_{,i} \neq 0) \quad (2.7)$$

This completes the proof of the theorem.

Theorem(2.2): Under the decomposition (2.1), the tensor fields R^h_{ijk}, R_{ij} and $Y_{j,k}$ satisfy the relation

$$\lambda_a R^h_{ijk} = \lambda_i R_{jk} - \lambda_j R_{ik} = X_{,i}\, Y_{j,k}\,. \quad (2.8)$$

Proof. With the help of (1.4), (1.12) and (1.13), we get

$$\lambda_a R^h_{ijk} = \lambda_i R_{jk} - \lambda_j R_{ik} \quad (2.9)$$

Multiplying (2.1) by λ_h and using relation (2.2), we have

$$\lambda_h\, R^h_{ijk} = X_{,i} Y_{j,k} \quad (2.10)$$

From (2.9) and (2.10), we get the required relation (2.8).

Theorem(2.3): Under the decomposition (2.1), the quantities λ_a and $p^{,h}$ behave like the recurrent vector and contravariant vector respectively. The recurrent form of these quantities are given by

$$\lambda_{a,m} = \mu_m \lambda_a\,, \quad (2.11)$$

and

$$P^{,h}_{,m} = -\mu_m P^{,h}\,. \quad (2.12)$$

Proof. Differentiating (2.9) covariantly w.r.t. x^m and using (2.1), we have

$$\lambda_{a,m} P^{,a} X_{,i} Y_{j,k} = \lambda_{i,m} R_{jk} - \lambda_{j,m} R_{ik} \quad (2.13)$$

Multiplying (2.13) by λ_a and using (2.1) and (2.9), we have

$$\lambda_{a,m}\,(\lambda_i R_{jk} - \lambda_j R_{ik}) = \lambda_a\,(\lambda_{i,m} R_{jk} - \lambda_{j,m} R_{ik})\,, \quad (2.14)$$

Now, multiplying (2.14) by λ_h, we have

$$\lambda_{a,m}\,(\lambda_i R_{jk} - \lambda_j R_{ik})\,\lambda_h = \lambda_a \lambda_h\,(\lambda_{i,m} R_{jk} - \lambda_{j,m} R_{ik}) \quad (2.15)$$

Since the expression on the right hand side of (2.15) is symmetric in a and h, therefore

$$\lambda_{a,m} \lambda_h = \lambda_{h,m} \lambda_a \quad (2.16)$$

provided $\lambda_i R_{jk} - \lambda_j R_{ik} \neq 0$.

The vector field λ_a being non-zero, we can choose a proportional vector μ_m such that

$$\lambda_{a,m} = \mu_m \lambda_a \quad (2.17)$$

Further, differentiating (2.2) covariantly w.r.to x^m and using (2.17), we have

$$\lambda_h P^{,h}_{,m} + P^{,h} \lambda_{h,m} = 0.$$

Making the use of equation (2.11), we have

$$P^{,h}_{,m} = -\mu_m P^{,h}\,, \quad (\text{since } \lambda_h \neq 0) \quad (2.18)$$

This proves the theorem.

Theorem (2.4): Under the decomposition (2.1), the curvature tensor and Holomorphically projective curvature tensor are equal if

$$Y_{k,m}\left\{\left(X_{,i}\delta^h_j - X_{,j}\delta^h_i\right) + X_{,l}\left(F^l_i F^h_j - F^l_j F^h_i\right)\right\} + 2X_{,l} F^l_i Y_{j,m} F^h_k = 0. \quad (2.19)$$

Proof. The Holomorphically Projective curvature tensor P^h_{ijk} is defined by

$$P^h_{ijk} = R^h_{ijk} + \frac{1}{(n+2)}\left(R_{ik}\partial^h_j - R_{jk}\partial^h_i + S_{ik} F^h_j - S_{jk} F^h_i + 2S_{ij} F^h_k\right) \quad (2.20)$$

where $S_{ij} = F^a_i R_{aj}$.

which may be expressed as

$$P^h_{ijk} = R^h_{ijk} + D^h_{ijk}\,, \quad (2.21)$$

where

$$D^h_{ijk} = \frac{1}{(n+2)}\left(R_{ik}\delta^h_j - R_{jk}\delta^h_i + S_{ik}F^h_j - S_{jk}F^h_i + 2S_{ij}F^h_k\right), \quad (2.22)$$

Contracting indices h and k in (2.1), we have

$$R_{ij} = P^{,k}X_{,i}Y_{j,k} \quad (2.23)$$

In view of (2.23), we have

$$S_{ij} = F^l_i P^{,m}X_{,l}Y_{j,m} \quad (2.24)$$

Making use of equations (2.23) and (2.24) in (2.22), we obtain

$$D^h_{ijk} = \frac{1}{(n+2)}\left[Y_{k,m}P^{,m}\left\{\left(X_{,i}\delta^h_j - X_{,j}\delta^h_i\right) + X_{,l}\left(F^l_iF^h_j - F^l_jF^h_i\right)\right\} + 2P^{,m}X_{,l}F^l_iY_{j,m}F^h_k\right] = 0. \quad (2.25)$$

From equation (2.21), it is clear that

$$P^h_{ijk} = R^h_{ijk} \quad \text{if} \quad D^h_{ijk} = 0,$$

which in view of (2.25) becomes

$$Y_{k,m}P^{,m}\left\{\left(X_{,i}\delta^h_j - X_{,j}\delta^h_i\right) + X_{,l}\left(F^l_iF^h_j - F^l_jF^h_i\right)\right\} + 2P^{,m}X_{,l}F^l_iY_{j,m}F^h_k = 0. \quad (2.26)$$

Multiplying (2.26) by λ_m and using (2.2), we obtain the required condition (2.19).

Theorem(2.5): Under the decomposition (2.1), the vector $X_{,i}$ and the tensor field $Y_{j,k}$ satisfy the relation

$$(\lambda_m + \mu_m)X_{,i}Y_{j,k} = X_{,i}Y_{j,km} + Y_{j,k}X_{i,m} \quad (2.27)$$

Proof. Differentiating (2.1) covariantly w.r.t. x^m and using (1.12), (2.1) and (2.12), we get the required result (2.27).

Theorem(2.6): Under the decomposition (2.1), the scalar curvature R, satisfy the relation

$$\lambda_k R = R_{,k} = g^{ij}X_{,i}Y_{j,k} \quad . \quad (2.28)$$

Proof. Contracting indices h and k in (2.1), we have

$$R_{ij} = P^{,k}X_{,i}Y_{j,k} \quad (2.29)$$

Multiplying (2.29) by g^{ij} on both sides, we have

$$g^{ij}R_{ij} = g^{ij}P^{,k}X_{,i}Y_{j,k} \quad (2.30)$$

or,

$$R = g^{ij}P^{,k}X_{,i}Y_{j,k} \;, \quad (2.31)$$

Now, multiplying (2.31) by λ_k and using (2.2), we have

$$\lambda_k R = g^{ij}X_{,i}Y_{j,k} \quad \text{or,} \quad R_{,k} = g^{ij}X_{,i}Y_{j,k} \;.$$

[by using (*1.14*)]

which completes the proof of the theorem.

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