Ultra Scientist Vol. 24(1)A, 1-8 (2012).

On bilateral generating functions involving modified Jacobi Polynomials from the group –theoretic view point

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(Acceptance Date 29th November, 2011)

Abstract

In this note, we have obtained some novel results on bilateral generating functions involving $P_{n+k}^{(\alpha+n,\ \beta)}(x)$, a modified form of Jacobi Polynomials by group theoretic method. In fact, in section 1, we have introduced a linear partial differential operator R which do not seem to have appeared in the earlier investigations and then we have obtained the extended form of the group generated by R. Finally, in section 2, we have obtained a novel generating relation involving the polynomial under consideration with the help of which, we have proved a general theorem on bilateral generating relations of $P_{n+k}^{(\alpha+n,\ \beta)}(x)$. Some particular cases of interest are also discussed.

Key wards : Bilateral generating relation, Jacobi Polynomials,

AMS-2000 classification code : 33C65.

1. Introduction

Special functions are the solutions of a wide class of mathematically and physically relevant functional equations. Generating functions play a large role in the study of special functions. There are various methods of obtaining generating functions. But it has been found that group-theoretic method of obtaining generating functions is much potent one in comparison to analytic method. The study of special functions, in particular, generating functions of special functions by group-theoretic method was originally introduced by L.Weisner[1] while studying generating functions of Hypergeometric function in the year 1955. From seventies and onwards (*i.e.* just after the publication of the monograph "obtaining generating functions" by E.B.McBride[2]) of the last century, Weisner's group theoretic method has been utilized by researchers while deriving generating functions of various special functions.

In the present article we have obtained some novel bilateral generating functions of $P_n^{(\alpha+n,\,\beta)}(x)$, a modification of Jacobi Polynomials, by group-theoretic method, where $P_n^{(\alpha,\,\beta)}(x)$ is defined[3] by

$$P_n^{(\alpha,\,\beta)}(x) = \frac{(1+\alpha)_n}{n!}\,{}_2F_1\left[\begin{matrix} -n,\ 1+\alpha+\beta+n\ ; & \\ & \dfrac{1-x}{2} \\ 1+\alpha\ ; & \end{matrix}\right] \tag{1.1}$$

The main result of our investigation is stated in the form of the following theorem. For previous works on bilateral generating functions of Jacobi /modified Jacobi Polynomials, one may refer to the works[4-8].

Theorem :

If there exists a unilateral generating relation of the form :

$$G(x,t) = \sum_{n=0}^{\infty} a_n\, P_{n+k}^{(\alpha+n,\,\beta)}(x)\, t^n, \tag{1.2}$$

then

$$(1+z)^{\alpha}\left\{1+\frac{z}{2}(1-x)\right\}^{-1-\alpha-\beta-k} G\left(\frac{x-\dfrac{z}{2}(1-x)}{1+\dfrac{z}{2}(1-x)},\ \frac{tz(1+z)}{\left\{1+\dfrac{z}{2}(1-x)\right\}^2}\right)$$

$$= \sum_{n=0}^{\infty} z^n \sigma_n(x,t), \tag{1.3}$$

where

$$\sigma_n(x,t) = \sum_{p=0}^{n} a_p \binom{n+k}{p+k} P_{n+k}^{(\alpha+n-2p,\beta)}(x)\, t^p\,. \tag{1.4}$$

2. *Derivation of the operator and the extended group :*

Let

$$R = R_1\frac{\partial}{\partial x} + R_2\frac{\partial}{\partial y} + R_3\frac{\partial}{\partial z} + R_0 \tag{2.1}$$

such that

$$R\left(P_{n+k}^{(\alpha+n,\,\beta)}(x)\, y^{\alpha} z^{n}\right) = a_n P_{n+k+1}^{(\alpha+n-1)}(x)\, y^{\alpha-2} z^{n+1} \tag{2.2}$$

where $R_i\ (i = 0, 1, 2, 3)$ are functions of x, y, z but independent of n, α and a_n is a function of n, α.

Noticing the following differential recurrence relation[3]

$$\frac{d}{dx}\left(P_n^{(\alpha,\,\beta)}(x)\right) = \frac{1}{1-x^2}\Big[(n+\alpha+\beta+1)(x-1) + 2\alpha\Big] P_n^{(\alpha,\,\beta)}(x) - 2(n+1) P_{n+1}^{(\alpha-1,\,\beta)}(x), \tag{2.3}$$

we define

$$R = (1-x^2)\,y^{-2}z\frac{\partial}{\partial x} - (x+1)\,y^{-1}z\frac{\partial}{\partial y} - 2xy^{-2}z^2\frac{\partial}{\partial z} - (1+\beta+k)(x-1)\,y^{-2}z \tag{2.4}$$

such that

$$R\left(P_{n+k}^{(\alpha+n,\,\beta)}(x)\, y^{\alpha} z^{n}\right) = -2(n+k+1) P_{n+k+1}^{(\alpha+n-1,\beta)}(x)\, y^{\alpha-2} z^{n+1}.$$

We now proceed to find the extended form of the group generated by R *i.e.* we shall find $e^{wR} f(x,y,z)$ where $f(x,y,z)$ is arbitrary function and w is arbitrary constant ,real or complex.

If $\phi(x,y,z)$ be a solution of $R\phi(x,y,z)=0$ and if we transform the Operator R to E such that

$$E = R_1 \frac{\partial}{\partial x} + R_2 \frac{\partial}{\partial y} + R_3 \frac{\partial}{\partial z}$$

then

$$E = \phi^{-1}(x,y,z) R \phi(x,y,z)$$

i.e. $R = \phi(x,y,z) E \phi^{-1}(x,y,z)$.

Therefore, we have

$$e^{wR} f(x,y,z) = e^{w\phi(x,y,z)E\phi^{-1}(x,y,z)} f(x,y,z)$$
$$= \phi(x,y,z) e^{wE}\left(\phi^{-1}(x,y,z) f(x,y,z)\right).$$

Finally, we choose new variables X, Y, Z so that the operator E is transformed into the operator $D = \frac{\partial}{\partial X}$. Under this change of variables, let $\phi^{-1}(x,y,z) f(x,y,z)$ be transformed into $F(x,y,z)$.

Therefore, by Taylor's theorem, we get

$$e^{wR} f(x,y,z) = \phi(x,y,z)\, e^{wD}\left(F(x,y,z)\right)$$
$$= \phi(x,y,z) F(X+w, Y, Z)$$
$$= \phi(x,y,z)\, g(x,y,z)$$

supposing that $F(X+w,Y,Z)$ is transformed into $g(x,y,z)$ by inverse substitution.

Let

$$\phi(x,y,z) = (1+x)^{-\beta-k}\, y z^{-1} \ ,$$

$$x = -\left(\frac{Z}{XY^2} + 1\right),$$

$$y = \left(\frac{2XY^2 + Z}{XY^3}\right),$$

$$z = -\left(\frac{Z}{X^2 Y^4} + \frac{2}{XY^2}\right).$$

Then by the method out lined, we find

$$e^{wR} f(x,y,z) = \left[1-(1-x)y^{-2}zw\right]^{-k-\beta-1}$$
$$\times\ f\left(\frac{x+(1-x)y^{-2}zw}{1-(1-x)y^{-2}zw}, \frac{y(1-2y^{-2}zw)}{1-(1-x)y^{-2}zw}, \frac{z(1-2y^{-2}zw)}{\{1-(1-x)y^{-2}zw\}^2}\right). \tag{2.6}$$

Now we shall obtain a novel generating relation of $P_{n+k}^{(\alpha+n,\ \beta)}(x)$ with the help of which the above stated theorem has been proved.

First , we notice that

$$e^{wR}\left(P_{n+k}^{(\alpha+n,\ \beta)}(x) y^{\alpha} z^{n}\right) = y^{\alpha} z^{n} \left[1-2y^{-2}zw\right]^{\alpha+n} \left[1-(1-x)y^{-2}zw\right]^{-1-\alpha-\beta-2n-k}$$
$$\times\ P_{n+k}^{(\alpha+n,\ n)}\left(\frac{x+(1-x)y^{-2}zw}{1-(1-x)y^{-2}zw}\right). \tag{2.7}$$

Again,

$$e^{wR}\left(P_{n+k}^{(\alpha+n,\ \beta)}(x)y^{\alpha}z^{n}\right)=\sum_{p=0}^{\infty}\frac{w^{p}}{p!}R^{p}\left(P_{n+k}^{(\alpha+n,\ \beta)}(x)y^{\alpha}z^{n}\right)$$

$$=\sum_{p=0}^{\infty}\frac{w^{p}}{p!}(-2)^{p}(n+k+1)_{p}\left(P_{n+k+p}^{(\alpha+n-p,\beta)}(x)y^{\alpha-2p}z^{n+p}\right) \tag{2.8}$$

$$=y^{\alpha}z^{n}\sum_{p=0}^{\infty}\frac{(n+k+1)_{p}}{p!}P_{n+k+p}^{(\alpha+n-p,\ \beta)}(x)(-2y^{-2}zw)^{p}\ .$$

Equating (2.7) and 2.8) and then replacing $-2y^{-2}zw$ by t, we get

$$(1+t)^{\alpha+n}\left[1+\frac{t}{2}(1-x)\right]^{-1-\alpha-\beta-k-2n}P_{n}^{(\alpha+n,\ \beta)}\left(\frac{x-\frac{t}{2}(1-x)}{1+\frac{t}{2}(1-x)}\right)$$

$$=\sum_{p=0}^{\infty}\frac{(n+k+1)_{p}}{p!}P_{n+k+p}^{(\alpha+n-p,\ \beta)}(x)t^{p}\ , \tag{2.9}$$

which does not seem to appear before .

Special cases :
Case 1 : Putting $k=0$, we get

$$(1+t)^{\alpha+n}\left[1+\frac{t}{2}(1-x)\right]^{-1-\alpha-\beta-2n}P_{n}^{(\alpha+n,\ \beta)}\left(\frac{x-\frac{t}{2}(1-x)}{1+\frac{t}{2}(1-x)}\right)$$

$$=\sum_{p=0}^{\infty}\frac{(n+1)_{p}}{p!}P_{n+p}^{(\alpha+n-p,\ \beta)}(x)t^{p}\ , \tag{2.10}$$

which is worthy of notice.

Case 2 : Putting $n=0$ in (2.10), we get

$$\sum_{p=0}^{\infty} P_p^{(\alpha-p,\,\beta)}(x)\,t^p = (1+t)^{\alpha}\left[1+\frac{t}{2}(1-x)\right]^{-1-\alpha-\beta},$$

which, with the help of the symmetry relation ;

$$P_n^{(\beta,\,\alpha)}(-x) = (-1)^n P_n^{(\alpha,\,\beta)}(x) \tag{2.11}$$

reduces to

$$\sum_{p=0}^{\infty} P_p^{(\alpha-p,\,\beta)}(x)\,t^p = (1-t)^{\beta}\left[1-\frac{t}{2}(1+x)\right]^{-1-\alpha-\beta} \tag{2.12}$$

and is the well known Feldhim's fourmula[9]. Relation (2.12) is also found derived[6,10,11] in.

We shall now prove the theorem-1 by using the generating relation (2.9).

Proof of the Theorem :

Now the right hand side of (1.3)

$$= \sum_{n=0}^{\infty} z^n \sigma_n(x,t)$$

$$= \sum_{n=0}^{\infty} z^n \sum_{p=0}^{n} a_p \binom{n+k}{p+k} P_{n+k}^{(\alpha+n-2p,\,\beta)}(x)\,t^p$$

[using (1.4)]

$$= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} z^{n+p}\, a_p \binom{n+p+k}{p+k} P_{n+p+k}^{(\alpha+n-p,\,\beta)}(x)\,t^p$$

$$= \sum_{p=0}^{\infty} a_p\,(tz)^p \sum_{n=0}^{\infty} \binom{n+p+k}{p+k} P_{n+k+p}^{(\alpha+n-p,\,\beta)}(x)\,z^n$$

$$= \sum_{p=0}^{\infty} a_p\,(tz)^p\,(1+z)^{\alpha+p}\left[1+\frac{z}{2}(1-x)\right]^{-1-\alpha-\beta-k-2p} P_{p+k}^{(\alpha+p,\,\beta)}\left(\frac{x-\frac{z}{2}(1-x)}{1+\frac{z}{2}(1-x)}\right)$$

[using (2.9)]

$$= (1+z)^{\alpha}\left[1+\frac{z}{2}(1-x)\right]^{-1-\alpha-\beta-k}\sum_{p=0}^{\infty} a_p \left[\frac{tz(1+z)}{\{1+\frac{z}{2}(1-x)\}^2}\right]^p P_{p+k}^{(\alpha+p,\,\beta)}\left(\frac{x-\frac{z}{2}(1-x)}{1+\frac{z}{2}(1-x)}\right)$$

$$= (1+z)^{\alpha}\left[1+\frac{z}{2}(1-x)\right]^{-1-\alpha-\beta-k} G\left(\frac{x-\frac{z}{2}(1-x)}{1+\frac{z}{2}(1-x)}\ ,\ \frac{tz(1+z)}{\{1+\frac{z}{2}(1-x)\}^2}\right)$$

[using(1.2)]

This completes the proof of the theorem.

Now we would like to point it out that theorem-1 can be proved as follows by the direct application of the operator R by using the method as discussed[4] in.

Let

$$G(x,t) = \sum_{n=0}^{\infty} a_n\ P_{n+k}^{(\alpha+n,\ \beta)}(x)\, t^n \quad . \tag{3.2}$$

Replacing t by tz in (3.2) and then multiplying both sides of the same by y^{α} , we get

$$y^{\alpha}\ G(x,tz) = \sum_{n=0}^{\infty} a_n \left(P_{n+k}^{(\alpha+n,\ \beta)}(x)\, y^{\alpha} z^n\right) t^n . \tag{3.3}$$

Now operating $(\exp wR)$ on both sides of (3.3), we get

$$(\exp wR)(y^{\alpha} G(x,tz)) = (\exp wR)\sum_{n=0}^{\infty} a_n \left(P_{n+k}^{(\alpha+n,\ \beta)}(x)\, y^{\alpha} z^n\right) t^n \tag{3.4}$$

The left member of (3.4), with the help of (2.6), becomes

$$y^{\alpha}(1-2y^{-2}zw)^{\alpha}\ \left[1-(1-x)y^{-2}zw\right]^{-1-\alpha-\beta-k} G\left(\frac{x+(1-x)y^{-2}zw}{1-(1-x)y^{-2}zw},\ \frac{tz(1-2y^{-2}zw)}{\{1-(1-x)y^{-2}zw\}^2}\right) \tag{3.5}$$

The right member of (3.4), with the help of (2.5), becomes

$$\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} a_n\, t^n \frac{w^p}{p!} R^p \left(P_{n+k}^{(\alpha+n,\,\beta)}(x) y^{\alpha} z^n\right) = \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} a_n\, t^n \frac{w^p}{p!} (-2)^p\, (n+k+1)_p\, P_{n+k+p}^{(\alpha+n-p,\,\beta)}(x) y^{\alpha-2p} z^{n+p}$$

$$= y^{\alpha} \sum_{n=0}^{\infty} z^{n+p} \sum_{p=0}^{\infty} a_n \frac{(n+k+1)_p}{p!} P_{n+k+p}^{(\alpha+n-p,\,\beta)}(x) \left(\frac{-2w}{y^2}\right)^p t^n$$

$$= y^{\alpha} \sum_{n=0}^{\infty} z^{n} \sum_{p=0}^{\infty} a_{n-p} \frac{(n-p+k+1)_p}{p!} P_{n+k}^{(\alpha+n-2p,\,\beta)}(x) \left(\frac{-2w}{y^2}\right)^p t^{n-p}\,.$$

Equating the above two results, we get

$$(1-2y^{-2}zw)^{\alpha}\ \left[1-(1-x)y^{-2}zw\right]^{-1-\alpha-\beta-k} G\left(\frac{x+(1-x)y^{-2}zw}{1-(1-x)y^{-2}zw},\ \ \frac{tz(1-2y^{-2}zw)}{\{1-(1-x)y^{-2}zw\}^2}\right)$$

$$= \sum_{n=0}^{\infty} z^n \sum_{p=0}^{n} a_p \frac{(p+k+1)_p}{(n-p)!} P_{n+k}^{(\alpha+n-2p,\,\beta)}(x) \left(\frac{-2w}{y^2}\right)^{n-p} t^p\,.$$

Putting $\dfrac{-2w}{y^2} = -1$, we get

$$(1+z)^{\alpha}\left[1+\frac{z}{2}(1-x)\right]^{-1-\alpha-\beta-k} G\left(\frac{x-\frac{z}{2}(1-x)}{1+\frac{z}{2}(1-x)}\ ,\ \frac{tz(1+z)}{\{1+\frac{z}{2}(1-x)\}^2}\right) = \sum_{n=0}^{\infty} z^n \sigma_n(x,t)\ ,$$

where

$$\sigma_n(x,t) = \sum_{p=0}^{n} a_p \binom{n+k}{p+k} P_{n+k}^{(\alpha+n-2p,\,\beta)}(x)\, t^p\,,$$

which is the theorem-1.

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