Ultra Scientist Vol. 24(1)A, 9-14 (2012).

Motions in a Riemannian space and some integrability conditions

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(Acceptance Date 29th November, 2011)

Abstract

Knebelman[1] studied and defined collineations and motions in generalised spaces. Levine[3] studied motions in linearly connected two dimensional spaces. Takano[2], studied on the existence of affine motion in a space with recurrent curvature tensor. Further, Negi and Rawat[6] studied affine motion in an almost Tachibana recurrent space. Rawat and Dobhal[9], studied on Projective motion in a Tachibana symmetric space .

In the present paper, we have studied Affine motion, Projective motion, Conformal motion, Integrability conditions of Killing equations also several theorems have been established and proved therein.

1. Motions in a Riemannian space.

Consider an n-dimensional Riemannian space V_n covered by a set of neighbourhoods with Coordinates ξ^x and endowed with the fundamental quadratic differential form

$$ds^2 = g_{\lambda x}(\xi)\, d\xi^\lambda\, d\xi^x \tag{1.1}$$

where the Greek indices x, λ, μ, ν, ..., run over the range 1, 2 ,3 , … n .

In the V_n referred to ξ^x, we consider a point transformation

$$T: \quad {}^*\xi^x = f^x(\xi^\nu); Det\left(\partial_\lambda f^x\right) \neq 0 \tag{1.2}$$

which establishes a one-to-one correspondence between the points of a region R and those of some other region *R, where ∂_λ stands for the partial derivation $\frac{\partial}{\partial \xi^\lambda}$.

During this point transformation, a point ξ^x in R is carried to a point $^*\xi^x$ in *R and a point $\xi^x + d\xi^x$ in R to a point $^*\xi^x + d^*\xi^x$ in *R .

If the distance d*s between two displaced points $^*\xi^x$ and $^*\xi^x + d^*\xi^x$ is always equal to the distance between the two original points ξ^x and $\xi^x+d\xi^x$ the point transformation (1.2) is called a motion or an isometry in the V_n.

(i) Affine motion in V_n: When a point transformation (1.2), transforms any pair of parallel vectors into a pair of parallel vectors, then (1.2) is called Affine motion in a V_n.

For an affine motion, we must have

$$\delta^m u^x(\,{}^*\xi\,) \stackrel{\text{def}}{=} d^m u^x(\,{}^*\xi) + \Gamma^x_{\mu\lambda}(\,{}^*\xi)\ \frac{m}{u}^{\lambda}(\,{}^*\xi)\, d^*\xi^\mu = 0\ . \tag{1.3}$$

(ii) Projective Motions in V_n: When a point transformation (1.2) transforms the system of geodesics into the same system, then (1.2) is called a projective Motion in V_n

The necessary and sufficient condition that (1.2) be a projective motion in a V_n is that the Lie-difference of $\Gamma^x_{\mu\lambda}$ with respect to (1.2) has the form

$${}^*\Gamma^x_{\mu\lambda} - \Gamma^x_{\mu\lambda} = A^x_\mu\, p_\lambda + A^x_\lambda\, p_\mu, \tag{1.4}$$

where p_λ is a covariant vector.

When (1.2) is an infinitesimal transformation ${}^*\xi^x = \xi^x + v^x(\xi)dt$, then the condition is

$$\underset{v}{\mathfrak{L}}\ \Gamma^x_{\mu\lambda} = A^x_\mu\, p_\lambda + A^x_\lambda\, p_\mu \tag{1.5}$$

(iii) Conformal Motion in V_n: When a point transformation (1.2) does not change the angle between two direction at a point, then (1.2) is called a conformal motion in V_n. The necessary and sufficient condition that (1.2) be conformal motion in a V_n is that the Lie –difference of $g_{\lambda x}$ with respect to (1.2) be proportional to $g_{\lambda x}$. [Schouten]

$${}'g_{\lambda x} - g_{\lambda x} = 2\,\phi\, g_{\lambda x} \tag{1.6}$$

Where ϕ is a scalar

When (1.2) is an infinitesimal transformation ${}^*\xi^x = f^x + v^x(\xi)dt$, then the condition is

$$\underset{v}{\mathfrak{L}}\, g_{\lambda x} = 2\,\phi\, g_{\lambda x}. \tag{1.7}$$

Now, we have the following theorem which is geometrically evident.

Theorem (1.1) : A motion in a V_n is an Affine motion.

Proof : To prove this, we apply the formula

$$\underset{v}{\mathfrak{L}}\nabla_\nu\, p^{x\lambda}_{\mu} - \nabla_\nu \underset{v}{\mathfrak{L}}\, p^{x\lambda}_{\dots\mu} = \left(\underset{v}{\mathfrak{L}}\ \Gamma^x_{\nu\rho}\right) P^{\rho\lambda}_{\dots\mu} + \left(\underset{v}{\mathfrak{L}}\ \Gamma^\lambda_{\nu\rho}\right) P^{x\rho}_{\dots\mu} - \left(\underset{v}{\mathfrak{L}}\ \Gamma^\rho_{\nu\mu}\right) P^{x\lambda}_{\dots\rho} \tag{1.8}$$

to the fundamental tensor $g_{\lambda x}$, we have

$$\underset{v}{\mathfrak{L}}\left(\nabla_\mu\, g_{\lambda x}\right) - \nabla_\mu\left(\underset{v}{\mathfrak{L}}\, g_{\lambda x}\right) = -\left(\underset{v}{\mathfrak{L}}\ \begin{Bmatrix}\rho\\ \mu\ \lambda\end{Bmatrix}\right) g_{\rho x} - \left(\underset{v}{\mathfrak{L}}\ \begin{Bmatrix}\rho\\ \mu\ x\end{Bmatrix}\right) g_{\lambda\rho}, \tag{1.9}$$

from which

$$\underset{v}{\mathfrak{L}}\ \begin{Bmatrix}x\\ \mu\ \lambda\end{Bmatrix} = \frac{1}{2}\, g^{x\rho}\left[\nabla_\mu \underset{v}{\mathfrak{L}}\ g_{\lambda\rho} + \nabla_\lambda \underset{v}{\mathfrak{L}}\ g_{\mu\rho} - \nabla_\rho \underset{v}{\mathfrak{L}}\ g_{\mu\lambda}\right]. \tag{1.10}$$

This equation shows that $\underset{v}{\mathfrak{L}}\ g_{\lambda x} = 0$ implies $\underset{v}{\mathfrak{L}}\ \begin{Bmatrix}x\\ \mu\ \lambda\end{Bmatrix} = 0$,

[**Note :** Under some global conditions $\underset{v}{\mathfrak{L}}\ \begin{Bmatrix}x\\ \mu\ \lambda\end{Bmatrix} = 0$ implies $\underset{v}{\mathfrak{L}}\ g_{\lambda x} = 0$.]

Theorem (1.2) : For a motion in a V_n the Lie-derivative of the curvature tensor and its successive covariant derivative Vanish.

Proof: To prove the above theorem, we apply the following

$$\nabla_\nu \underset{v}{\mathfrak{L}}\, \Gamma^x_{\mu\lambda} - \nabla_\mu \underset{v}{\mathfrak{L}}\, \Gamma^x_{\nu\lambda} + 2\, S_{\nu\mu}^{\cdots\rho} \underset{v}{\mathfrak{L}}\, \Gamma^x_{\rho\lambda} = \underset{v}{\mathfrak{L}}\, R_{\nu\mu\lambda}^{\cdots x} \qquad (1.11)$$

to the Christoffel symbol, we have

$$\nabla_\nu \underset{v}{\mathfrak{L}} \begin{Bmatrix} x \\ \mu\ \lambda \end{Bmatrix} - \nabla_\mu \underset{v}{\mathfrak{L}} \begin{Bmatrix} x \\ \nu\ \lambda \end{Bmatrix} = \underset{v}{\mathfrak{L}}\, K_{\nu\mu\lambda}^{\cdots x}.$$

Where $K_{\nu\mu\lambda}^{\cdots x}$ is the curvature tensor of V_n. Thus for a motion, we have

$$\underset{v}{\mathfrak{L}}\, K_{\nu\mu\lambda}^{\cdots x} = 0\ . \qquad (1.12)$$

On the other hand since a motion is an affine motion, the covariant derivation and the Lie-derivation are Commutative. Thus from (1.12), we obtain

$$\underset{v}{\mathfrak{L}}\, \nabla_\omega\, K_{\nu\mu\lambda}^{\cdots x} = 0, \underset{v}{\mathfrak{L}}\, \nabla_{\omega_2} \nabla_{\omega_1} K_{\nu\mu\lambda}^{\cdots x} = 0, \dots .$$

This proves the theorem.

2. Theorems on Projectively or Conformally related spaces:

Consider two Riemannian spaces V_n and V_n^* which are in geodesic correspondence. Then denoting the Christoffel symbols of them by $\begin{Bmatrix} x \\ \mu\ \lambda \end{Bmatrix}$ and $\begin{Bmatrix} x \\ \mu\ \lambda \end{Bmatrix}^*$ respectively, we have

$$\begin{Bmatrix} x \\ \mu\ \lambda \end{Bmatrix}^* = \begin{Bmatrix} x \\ \mu\ \lambda \end{Bmatrix} + A^x_\mu\, p_\lambda + A^x_\lambda\, p_\mu .$$

But since V_n and V_n^* are both Riemannian, the vector p_λ should be a gradient. Thus putting

$p_\lambda = \frac{1}{2}\, \partial_\lambda \log\phi$, we have

$$\begin{Bmatrix} x \\ \mu\ \lambda \end{Bmatrix}^* = \begin{Bmatrix} x \\ \mu\ \lambda \end{Bmatrix} + \frac{1}{2} A^x_\mu\, \partial_\lambda \log\phi + \frac{1}{2} A^x_\lambda\, \partial_\mu \log\phi\ . \qquad (2.1)$$

We now assume that the V_n admits a motion with symbol $\underset{v}{\mathfrak{L}} f$. Then, we have

$$\underset{v}{\mathfrak{L}}\, g_{\lambda x} = \nabla_\lambda v_x + \nabla_x v_\lambda = \partial_\lambda v_x + \partial_x v_\lambda - 2 \begin{Bmatrix} \rho \\ \lambda\ x \end{Bmatrix} v_\rho = 0\ .$$

Consequently on utilizing (2.1), we have

$$\underset{v}{\mathfrak{L}}\, g_{\lambda x} = \partial_\lambda v_x + \partial_x v_\lambda - 2 \left[\begin{Bmatrix} \rho \\ \lambda\ x \end{Bmatrix}^* - \frac{1}{2} A^\rho_\lambda\, \partial_x \log\phi - \frac{1}{2} A^\rho_x\, \partial_\lambda \log\phi \right] v_\rho$$

$$= \phi^{-1} \left[\partial_\lambda(\phi\, v_x) + \partial_x(\phi\, v_\lambda) - 2 \begin{Bmatrix} \rho \\ \lambda\ x \end{Bmatrix}^* \phi\, v_\rho \right]$$

Thus, denoting by $\overset{*}{g}_{\lambda x}$ the fundamental tensor of V_n^* and $\overset{*}{\underset{v}{\mathfrak{L}}} f$ the symbol defined by $\phi\, v_x$ in V_n^*, we have

$$\underset{v}{\mathfrak{L}}\, g_{\lambda x} = \phi^{-1} \overset{*}{\underset{v}{\mathfrak{L}}}\, \overset{*}{g}_{\lambda x}$$

Thus, we have.

Theorem (2.1): If two Riemannian spaces V_n and V_n^* are in geodesic correspondence and if V_n admits a group of motions, V_n^* also admits a group of motions.

Theorem (2.2): If a V_n admits a G_r of motions such that the rank of $\underset{a}{v}{}^x$ in a neighborhood is equal to $r < n$, then there exist $n - r$ $V_n^{,s}$, Corresponding to *n-r* independent solutions of $\underset{v}{\mathfrak{L}}\ \rho^2 = 0$, Which are Conformal to the given V_n and admit the same group as a group of motions.

Proof: Consider a V_n which admits an r-parameter group G_r of motions such that the rank of $\underset{a}{v}^{\ x}$ is in a certain neighborhood equal to $r < n$. Then we have $\underset{v}{\mathfrak{L}}\ g_{\lambda x} = 0$. In order that a space V_n^* Conformal to V_n admit the same group G_r as a group of motions, it is necessary and sufficient that there exist a function ρ^2 such that $\underset{v}{\mathfrak{L}}\ (\rho^2 g_{\lambda x}) = 0$ or $\underset{v}{\mathfrak{L}}\ \rho^2 = 0$. But on the other hand

$$\left(\underset{c}{\mathfrak{L}}\ \underset{b}{\mathfrak{L}}\right)\rho^2 = C_{cb}^{a}\ \underset{a}{\mathfrak{L}}\ \rho^2$$

and consequently $\underset{a}{\mathfrak{L}}\ \rho^2 = 0$ admits (n – r) independent solutions.

Note: The a-rank of the $\underset{a}{\mathfrak{L}}\ g_{\lambda x}$ is the rank of the matrix $\underset{a}{\mathfrak{L}}\ g_{\lambda x}$ where a denotes the rows and $\lambda\ x$ denotes the columns.

Theorem (2.3): In order that a G_r in X_n such that the rank of $\underset{a}{v}^{\ x}$ in a neighborhood is equal to $r \leq n$, can be regarded as a group of motions in a C_n, it is necessary and sufficient that the group be a subgroup of a group of Conformal transformations.

Proof: The necessity is evident. Conversely, if the group is a subgroup of a group of conformal transformations, it is a group of Conformal motions in a C_n (*i.e.* C_n stands for a Conformally Euclidean space). Consequently, there exists a V_n which is conformal to C_n and is itself a C_n which admits the group as a group of motions.

3. Integrability Conditions of Killing's equation :

The Integrability Conditions of killing's equation

$$\underset{v}{\mathfrak{L}}\ g_{\lambda x} = \nabla_\lambda v_x + \nabla_x v_\lambda = 0 \tag{3.1}$$

can be deduced from it, considering first the equation

$$\underset{v}{\mathfrak{L}}\ \left\{ {x \atop \mu\ \lambda} \right\} = \nabla_\mu \nabla_\lambda v^x + K_{\nu\mu\lambda}^{\cdots x}\, v^\nu = 0 \tag{3.2}$$

and next the mixed system of partial differential equations

$$\left.\begin{array}{ll} v_{\lambda x} + v_{x\lambda} = 0 & \left(v_{\lambda x} = v_{\lambda}^{\cdot\rho} g_{\rho x}\right) \\ \nabla_\lambda v^x = v_{\lambda}^{\cdot x}, & \nabla_\mu v_{\lambda}^{\cdot x} = -K_{\nu\mu\lambda}^{\cdots x}\, v^\nu, \end{array}\right\} \tag{3.3}$$

We know that the equations (3.1) and (3.2) or the equation (3.3) have for Integrability conditions

$$\underset{v}{\mathfrak{L}}\ K_{\nu\mu\lambda}^{\cdots x} = 0\,,\quad \underset{v}{\mathfrak{L}}\ \nabla_\omega\ K_{\nu\mu\lambda}^{\cdots x} = 0\,,$$

$$\underset{v}{\mathfrak{L}}\ \nabla_{\omega_2} \nabla_{\omega_1} K_{\nu\mu\lambda}^{\cdots x} = 0, \ \ldots\ldots\ldots\ldots, \tag{3.4}$$

4. Theorems on Affine and Projective motions :

We consider an A_n which admits a G_r of affine motions with the infinitesimal operators $\underset{a}{\mathfrak{L}}\ f = \underset{a}{v}^{\ \mu}\, \partial_\mu f$ such that the rank of $\underset{a}{v}{}^{x}$ in a neighborhood is $r \leq n$. Then, we have

$$\underset{a}{\mathfrak{L}}\ \Gamma^{x}_{\mu\lambda} = 0 \tag{4.1}$$

In order that an A^{*}_{n}, projectively related to the A_n, admits the G_r as a group of affine motions, it is necessary and sufficient that there exist a covariant vector field p_{λ} such that

$$\underset{a}{\mathfrak{L}}\ \left(\Gamma^{x}_{\mu\lambda} + {}_{\mu}\, A^{x}_{\lambda} + {}_{\lambda}\, A^{x}_{\mu} \right) = 0$$

Or

$$\underset{a}{\mathfrak{L}}\ {}_{\lambda} = 0 \,.$$

But this system of partial differential equation is completely integrable, hence

Theorem (4.1): If an A_n admits a G_r of affine motions with the infinitesimal operators $\underset{a}{\mathfrak{L}}\, f = \underset{a}{v}{}^{\mu}\, \partial_{\mu} f$ such that the rank of $\underset{a}{v}{}^{x}$ in a neighborhood is $r \leq n$, there exists always in A^{*}_{n} which is (not trivially) projectively related to A_n and which admits the same G_r as a group of affine motions.

We next consider an A_n which admits a G_r of projective motions such that the rank of $\underset{a}{v}{}^{x}$ in a neighborhood is $r \leq n$, Then, we have

$$\underset{a}{\mathfrak{L}}\ \Gamma^{x}_{\mu\lambda} = \underset{a}{{}_{\mu}}\ \mathrm{A}^{x}_{\lambda} + \underset{a}{{}_{\lambda}}\ A^{x}_{\mu}\,. \tag{4.3}$$

In order that an A^{*}_{n}, projectively related to the A_n, admit the same G_r as a group of affine motions , it is necessary and sufficient that there exist a covariant vector field p_{λ} such that

$$\underset{a}{\mathfrak{L}}\ \left(\Gamma^{x}_{\mu\lambda} + {}_{\mu}\, A^{x}_{\lambda} + {}_{\lambda}\, A^{x}_{\mu} \right) = 0 \tag{4.4}$$

Or

$$\underset{a}{\mathfrak{L}}\ {}_{\lambda} = - \underset{a}{{}_{\lambda}}\ . \tag{4.5}$$

On the other hand, substuting (4.3) in the identity

$$\left(\underset{c}{\mathfrak{L}}\ \underset{b}{\mathfrak{L}}\ \Gamma^{x}_{\mu\lambda} \right) = C^{a}_{cb}\ \underset{a}{\underbrace{\mathfrak{L}}}\ \Gamma^{x}_{\mu\lambda}\text{, we get}$$

$$\underset{c}{\mathfrak{L}}\ \underset{b}{{}_{\lambda}} - \underset{b}{\mathfrak{L}}\ \underset{c}{{}_{\lambda}} = C^{a}_{cb}\ \underset{a}{{}_{\lambda}}\,.$$

Which shows that r covariant vectors p_{λ} form a complete system with respect to G_r. Thus (4.5) is completely integrable and we have

Theorem (4.2) : when an A_n admits a G_r of projective motions such that the rank of v^{x} is $r \leq n$, there exists an A^{*}_{n} which is (not trivially) projectively related to A_n and which admits the same G_r as a group of affine motions.

From this, we obtain

Theorem (4.3) : In order that a G_r is an X_n such that the rank of $\underset{a}{v}{}^{x}$ is $r \leq n$, can be regarded as a group of affine motions in a D_n, it is necessary and sufficient that the G_r be a subgroup of the ordinary projective group.

Proof : The necessity is evident. Conversely, if the group G_r is a subgroup of the ordinary projective group, it is a group of projective motions in a D_n . consequently according to Theorem(4.2), there exist an A_n which is projectively related to D_n, and is itself a D_n and admits G_r as a group of affine motions. Thus, theorem (4.3) is proved.

5. Integrability condition of $\underset{v}{\mathfrak{L}} \; \Gamma^{x}_{\mu\lambda} = 0$.

We consider the Integrability condition of $\underset{v}{\mathfrak{L}} \; \Gamma^{x}_{\mu\lambda} = 0$, which can be written as

$$\left.\begin{aligned} \nabla_\lambda v^x &= v^{\cdot x}_{\lambda} - 2\, S^{\cdots x}_{\mu\lambda}\, v^\mu , \\ \nabla_\lambda v^{\cdot x}_{\lambda} &= -R^{\cdots x}_{\nu\mu\lambda}\, v^\nu , \end{aligned}\right\} \qquad (5.1)$$

Now from the following equations.

$$^{*}\mathrm{R}^{\cdots x}_{\nu\mu\lambda} = R^{\cdots x}_{\nu\mu\lambda} + \left(\nabla_\nu \underset{v}{\mathfrak{L}}\, \Gamma^{x}_{\mu\lambda} - \nabla_\mu \underset{v}{\mathfrak{L}}\, \Gamma^{x}_{\nu\lambda} + 2S^{\cdots\rho}_{\nu\mu} \underset{v}{\mathfrak{L}} \; \Gamma^{x}_{\rho\lambda} \right) dt .$$

and

$$\nabla_\nu \underset{v}{\mathfrak{L}}\, \Gamma^{x}_{\mu\lambda} - \nabla_\mu \underset{v}{\mathfrak{L}}\, \Gamma^{x}_{\nu\lambda} + 2S^{\cdots\rho}_{\nu\mu} \underset{v}{\mathfrak{L}} \; \Gamma^{x}_{\rho\lambda} = \underset{v}{\mathfrak{L}}\, R^{\cdots x}_{\nu\mu\lambda},$$

we have

$$\underset{v}{\mathfrak{L}}\, S^{\cdots x}_{\mu\lambda} = 0, \quad \underset{v}{\mathfrak{L}}\, R^{\cdots x}_{\nu\mu\lambda} = 0 , \qquad (5.2)$$

respectively. Then applying the formula

$$\underset{v}{\mathfrak{L}} \; \nabla_\nu \; P^{x\lambda}_{\cdots\mu} - \nabla_\nu \underset{v}{\mathfrak{L}} \; P^{x\lambda}_{\cdots\mu} = \left(\underset{v}{\mathfrak{L}}\, \Gamma^{x}_{\nu\lambda} \right) P^{\rho\lambda}_{\cdots\mu} + \left(\underset{v}{\mathfrak{L}}\, \Gamma^{\lambda}_{\nu\rho} \right) P^{x\rho}_{\cdots\mu} - \left(\underset{v}{\mathfrak{L}}\, \Gamma^{\rho}_{\nu\mu} \right) P^{x\lambda}_{\cdots\rho} .$$

to $S^{\cdots x}_{\mu\lambda}$ and $R^{\cdots x}_{\nu\mu\lambda}$, we obtain

$$\underset{v}{\mathfrak{L}} \; \nabla_\nu \; P^{\cdots x}_{\mu\lambda} = 0, \; \underset{v}{\mathfrak{L}} \; \nabla_\omega \; R^{\cdots x}_{\nu\mu\lambda} = 0 . \qquad (5.3)$$

respectively.

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