Ultra Scientist Vol. 24(1)A, 15-18 (2012).

# Common fixed point theorems for two mappings in fuzzy metric space

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**(Acceptance Date 25th December, 2011)**

## Abstract

The concept of fuzzy sets was introduced initially by Zadeh in 1965. Georgeand Veeramani[3] modified the concept of fuzzy metric space introduced[5]. Fisher[2], Aliouche and Fisher[1], Telci[10] proved some related fixedpoint theorems in compact metric spaces. Recently, Rao *et. al*.[7] proved[8,9] some fixed point theorems in sequentially compact fuzzy metric spaces.Motivated by a work due to Popa[6], we have observed that proving fixed point theorems using an implicit relation is a good idea since it covers several contractive conditions rather than one contractive condition.

## Introduction

*Theorem-1:* Let T & P be two self mappings of a complete Fuzzy Metric Space (X, S, *) with t - norm * defined by a * b = min {a, b} : a, b $\in$ [0,1] satisfying the conditions

1. $S(Tx, Py, z, kt) \geq \min\{ S(x, y, z, t), \frac{S(x, Tx, z, t)\ S(y, Py, z, t)}{S(x, y, z, t)}, \frac{S(x, y, z, t) + S(x, Tx, z, t)}{S(x, y, z, t)}\}$ for all x, y,z in X and $0 < k < 1$, $t > 0$

2. $S(x, y, z, t) \to 1$ as $t \to \infty$ Then T & P have a unique common fixed point.

*Proof:* Consider an arbitrary point $x_0$ in X and define a sequence $\{x_n\}$ in X by $x_{2n+1} = Tx_{2n}$, $x_{2n+2} = Px_{2n+1}$ for all n= 0,1,2,……… On using (1) for any $p \in N$, we have

$S(x_1, x_2, x_p, kt) = S(Tx_0, Px_1, x_p, kt)$ $\geq \min\{S(x_0, x_1, x_p, t),$

$$\frac{S(x_0, Tx_0, x_p, t)\, S(x_1, Px_1, x_p, t)}{S(x_0, x_1, x_p, t)},$$

$$\frac{S(x_0, x_1, x_p, t) + S(x_0, Tx_0, x_p, t)}{S(x_0, x_1, x_p, t)}\}$$

$\geq \min\{S(x_0, x_1, x_p, t),$

$$\frac{S(x_0, x_1, x_p, t)\, S(x_1, x_2, x_p, t)}{S(x_0, x_1, x_p, t)},$$

$$\frac{S(x_0, x_1, x_p, t) + S(x_0, x_1, x_p, t)}{S(x_0, x_1, x_p, t)}\}$$

$\geq \min\{S(x_0, x_1, x_p, t), S(x_1, x_2, x_p, t)\}$

This implies that

$S(x_1, x_2, x_p, kt)\ S(x_0, x_1, x_p, t)$

Again using (2.1.1) for any $p \in N$ we have

$S(x_2, x_3, x_p, kt) = S(Px_1, Tx_2, x_p, kt)$

$= S(Tx_2, Px_1, x_p, kt)$

$\geq \min\{ S(x_2, x_1, x_p, t),$

$$\frac{S(x_2, Tx_2, x_p, t)\, S(x_1, Px_1, x_p, t)}{S(x_1, x_2, x_p, t)},$$

$$\frac{S(x_1, x_2, x_p, t) + S(x_2, Tx_2, x_p, t)}{S(x_1, x_2, x_p, t)}\}$$

$\geq \min\{S(x_1, x_2, x_p, t),$

$$\frac{S(x_2, x_3, x_p, t)\, S(x_1, x_2, x_p, t)}{S(x_1, x_2, x_p, t)},$$

$$\frac{S(x_1, x_2, x_p, t) + S(x_2, x_3, x_p, t)}{S(x_1, x_2, x_p, t)}\}$$

$\geq \min\{S(x_2, x_3, x_p, t), S(x_1, x_2, x_p, t)\}$

This implies that

$S(x_2, x_3, x_p, kt) \geq S(x_1, x_2, x_p, t)$

Inductively we have

$S(x_n, x_{n+1}, x_p, kt) \geq S(x_{n-1}, x_n, x_p, t)$

$\geq S(x_{n-2}, x_{n-1}, x_p, t/k)$

$\geq \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\geq \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\geq S(x_0, x_1, x_p, t/k^{n-1})$

Or

$S(x_n, x_{n+1}, x_p, kt) \geq S(x_0, x_1, x_p, t/k^n)$

So for $p, q \in N$ & $t > 0$ we have for $k = 3$

$S(x_n, x_{n+p}, x_{n+p+q}, 3t) \geq S(x_n, x_{n+1}, x_{n+p+q}, t) * S(x_n, x_{n+1}, x_{n+p}, t)$

$* S(x_{n+1}, x_{n+p}, x_{n+p+q}, t)$

$\geq S(x_0, x_1, x_{n+p+q}, t/k^n) * S(x_0, x_1, x_{n+p}, t/k^n)$
$* S(x_{n+1}, x_{n+2}, x_{n+p+q}, t/3) * S(x_{n+2}, x_{n+p}, x_{n+p+q}, t/3)$

$\geq S(x_0, x_1, x_{n+p+q}, t/k^n) * S(x_0, x_1, x_{n+p}, t/k^n)$
$* S(x_0, x_1, x_{n+p+q}, t/3k^{n+1}) * S(x_0, x_1, x_{n+p}, t/3k^{n+1})$
$* S(x_{n+2}, x_{n+p}, x_{n+p+q}, t/3)$

$\geq S(x_0, x_1, x_{n+p+q}, t/k^n) * S(x_0, x_1, x_{n+p}, t/k^n)$
$* S(x_0, x_1, x_{n+p+q}, t/3k^{n+1}) * S(x_0, x_1, x_{n+p}, t/3k^{n+1})$
$* S(x_{n+2}, x_{n+p}, x_{n+p+q}, t/3)$ ……….

$\geq S(x_0, x_1, x_{n+p+q}, t/k^n) * S(x_0, x_1, x_{n+p}, t/k^n)$
$* S(x_0, x_1, x_{n+p+q}, t/3k^{n+1}) * S(x_0, x_1, x_{n+p}, t/3k^{n+1}) * \ldots * S(x_0, x_1, x_{n+p+q}, t/k^{n+p-2}3^{p-2}) * S(x_0, x_1, x_{n+p}, t/k^{n+p-2}3^{p-2})$
$* S(x_{n+p-1}, x_{n+p}, x_{n+p+q}, t/3^{p-2})$

$\geq S(x_0, x_1, x_{n+p+q}, t/k^n) * S(x_0, x_1, x_{n+p}, t/k^n)$
$* S(x_0, x_1, x_{n+p+q}, t/3k^{n+1}) * S(x_0, x_1, x_{n+p}, t/3k^{n+1}) * \ldots * S(x_0, x_1, x_{n+p+q}, t/k^{n+p-2}3^{p-2}) * S(x_0, x_1, x_{n+p}, t/k^{n+p-2}3^{p-2})$

$$* S(x_0, x_1, x_{n+p+q}, t/3^{p-1}k^{n+p-1})$$

On taking $\lim n \to \infty$ we have

$\lim_{n\to\infty} S(x_n, x_{n+p}, x_{n+p+q}, 3t) \geq 1*1*1*$…….. (2p-1) times

Which implies that

$$S(x_n, x_{n+p}, x_{n+p+q}, 3t) \to 1 \text{ as } n \to \infty$$

*i.e.* for even $\varepsilon > 0$, $t > 0$, $n \exists\ n_0 \in N$ such that $S(x_n, x_{n+p}, x_{n+p+q}, t) > 1-\varepsilon$ for all $n \geq n_0$

Thus $\{x_n\}$ is a Cauchy Sequence. By the completeness of the space, there is a point u in X such that

$\lim_{n\to\infty} x_n = u$

Now we shall prove that u is a fixed point of T By (2.1.1) we have

$$S(Tu, x_{2n+2}, x_n, kt) \geq S(Tu, Px_{2n+1}, x_n, kt)$$
$$\geq \min\{S(u, x_{2n+1}, x_n, t),$$
$$\frac{S(u, Tu, x_n, t)\, S(x_{2n+1}, Px_{2n+1}, x_n, t)}{S(u, x_{2n+1}, x_n, t)},$$
$$\frac{S(u, x_{2n+1}, x_n, t) + S(u, Tu, x_n, t)}{S(u, x_{2n+1}, x_n, t)}\}$$

on taking $\lim_{n\to\infty}$ we have

$$S(Tu, u, u, kt) \geq \min\{S(u, Tu, u, t), 1\}$$
$$= S(u, Tu, u, t)$$

Which yields $Tu = u$

Similarly we can prove $Pu = u$

Thus $Tu = u = Pu$

Hence u is a common fixed point of T & P. For the uniqueness of u, let v be another common fixed point of T & P. Then by (2.1.1) we have

$$S(u, u, v, kt) = S(Tu, Pu, v, kt)$$
$$\geq S(u, u, v, t)$$

which gives us $u = v$

To prove T & P are continuous at u. Let $\{y_n\}$ be a sequence in X such that $\lim_{n\to\infty} y_n = u$

On using (2.1.1) we have

$$S(Ty_n, x_{2n+2}, x_m, kt) = S(Ty_n, Px_{2n+1}, x_m, kt)$$
$$\geq \min\{S(y_n, x_{2n+1}, x_m, t),$$
$$\frac{S(y_n, Ty_n, x_m, t)\, S(x_{2n+1}, Px_{2n+1}, x_m, t)}{S(y_n, x_{2n+1}, x_m, t)},$$
$$\frac{S(y_n, x_{2n+1}, x_m, t) + S(y_n, Ty_n, x_m, t)}{S(y_n, x_{2n+1}, x_m, t)}\}$$

On taking $\lim_{n\to\infty}$ or $_{m\to\infty}$ we have

$$S(\lim Ty_n, u, u, kt) \geq \min\{1, S(u, \lim Ty_n, u, t)\}$$

This implies that

$$\lim_{n\to\infty} Ty_n = u = Tu = T \lim_{n\to\infty} y_n$$

Hence T is continuous at u, Similarly we can show that P is continuous at u.

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