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A Note on the form of Wave Equation used in Harish-Chandra's Paper 'Motion of an Electron in the Field of a Magnetic Pole'

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Abstract

In this paper we have tried to go through the research paper[1] of Prof. Harish-Chandra with the intention of getting some idea of method of thinking and working of that great mathematician as well as of his Ph.D. thesis supervisor Prof. P.A.M. Dirac. The wave equation for the motion of an electron derived by Dirac[2,3] has been transformed to the form suitable to the problem under consideration by Harish-Chandra. It is interesting and enjoying to clarify the steps directly written by him. For us it is a great job to fill the gaps to verify his steps. This paper is consequence of our effort in this direction.

Key words: Wave function, Hamiltonian, Wave equation, Pauli operators, Magnetic pole, Metric tensor.

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Introduction

1. Problem under consideration :

The dynamical system considered in his paper[1] is the motion of an electron moving in the field of a magnetic pole. The magnetic pole is supposed to be at the origin of reference and the electron of charge $-e$ is in the vicinity of this pole. We know that the motion of a dynamical system is governed by the Schrödinger's wave equation.

2. Mathematical formulation :

Schrödinger's wave equation[2] pp.111 is

$$i\hbar \frac{\partial}{\partial t}\Psi(x,y,z,t) = H\Psi(x,y,z,t), \qquad (1)$$

where H is energy operator *i.e.* Hamiltonian and $\Psi(x, y, z, t)$ is wave function. If we assume that

$$\Psi(x, y, z, t) = \psi(x, y, z)\eta(t),$$

then $i\hbar\psi \dfrac{d\eta}{dt} = H(\psi\eta) = \eta H\psi$,

i.e. $\quad i\hbar \dfrac{1}{\eta}\dfrac{d\eta}{dt} = \dfrac{H\psi}{\psi} = E$,

where E is a constant called some eigenvalue of operator H and so we have

$$\frac{1}{\eta} d\eta = -\frac{iE}{\hbar} dt \quad \text{and} \quad H\psi = E\psi ,$$

i.e. $\quad \eta = Ce^{-\frac{iE}{\hbar}t}$ and $H\psi = E\psi$, (2)

hence $\Psi(x, y, z, t)$ will be known if $\psi(x, y, z)$ satisfying $H\psi = E\psi$ is known.

For simplicity he took the velocity of light equal to 1 and the Plank's constant h equal to 2π which implies $\hbar$ equal to 1, also using the notation in cartesian coordinate system $(x, y, z) = (x^1, x^2, x^3)$ and spherical polar coordinate system $(r, \theta, \varphi) = (\xi^1, \xi^2, \xi^3)$. He assumed the strength of the magnetic pole to be $\dfrac{n}{2e}$ and then obtained the suitable vector potential $\mathbf{A} = (A_1, A_2, A_3) = (A_r, A_\theta, A_\varphi)$ whose component are $A_r = 0$, $A_\theta = 0$ and $A_\varphi = \dfrac{n}{2e}(1 - \cos\theta)$. The Hamiltonian H for the electron[2] (pp-257) suitable to our problem may be written as

$$H = -\rho_1 \boldsymbol{\sigma}.(\mathbf{p} + \frac{e}{c}\mathbf{A}) - \rho_3 \mu c , \tag{3}$$

where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, $\mathbf{p} = -i\hbar\left(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z}\right)$ is the momentum and μ the mass of electron. σ_1, σ_2, σ_3 and ρ_1, ρ_2, ρ_3 are the quantities called two independent sets of Pauli operators[2,3,5] which satisfy the relations

$$\begin{cases} \sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij} , \ \rho_i\rho_j + \rho_j\rho_i = 2\delta_{ij} \\ \sigma_i\rho_j = \rho_j\sigma_i , \ \sigma_1\sigma_2\sigma_3 = i , \ \rho_1\rho_2\rho_3 = i \end{cases}$$

$$(i, j = 1, 2, 3) \tag{4}$$

3. Reduction of the Hamiltonian to polar form :

By equation (3) Hamiltonian of electron can be written as

$H = -\rho_1 \boldsymbol{\sigma}.(\mathbf{p} + e\mathbf{A}) - \rho_3\mu = -\rho_1 \ (\boldsymbol{\sigma}.\mathbf{p}) - e\rho_1 (\boldsymbol{\sigma}.\mathbf{A}) - \rho_3\mu$,

$$= -\rho_1(\boldsymbol{\sigma}.\mathbf{p}) - e\rho_1(\sigma_1 A_1 + \sigma_2 A_2 + \sigma_3 A_3) - \rho_3\mu,$$

$$= -\rho_1(\boldsymbol{\sigma}.\mathbf{p}) - e\rho_1(\sigma^1 g_{11} A_1 + \sigma^2 g_{22} A_2 + \sigma^3 g_{33} A_3) - \rho_3\mu ,$$

$$= -\rho_1(\boldsymbol{\sigma}.\mathbf{p}) - e\rho_1\left(\sum_{k=1}^{3} \sigma^k A_k\right) - \rho_3\mu$$

$$= -\rho_1(\boldsymbol{\sigma}.\mathbf{p}) - e\rho_1\left(\sum_{\alpha} \sigma^\alpha A_\alpha\right) - \rho_3\mu ,$$

$$= -\rho_1(\boldsymbol{\sigma}.\mathbf{p}) - e\rho_1(\sigma^r A_r + \sigma^\theta A_\theta + \sigma^\varphi A_\varphi) - \rho_3\mu$$

$$= -\rho_1(\boldsymbol{\sigma}.\mathbf{p}) - e\rho_1(\sigma^\varphi A_\varphi) - \rho_3\mu ,$$

$$= -\rho_1(\boldsymbol{\sigma}.\mathbf{p}) - e\rho_1\left(\sigma_\varphi g^{\varphi\varphi} A_\varphi\right) - \rho_3\mu$$

$$= -\rho_1(\boldsymbol{\sigma}.\mathbf{p}) - e\rho_1\left(\sigma_\varphi \frac{1}{r^2\sin^2\theta} A_\varphi\right) - \rho_3\mu .$$

Thus we have

$$H = -\rho_1(\boldsymbol{\sigma}.\mathbf{p}) - \rho_3\mu - \rho_1 \frac{\sigma_\varphi}{r^2\sin^2\theta}\frac{n}{2}(1-\cos\theta), \tag{5}$$

Also here we use metric tensor[4] g_{ij} or $g_{\alpha\beta}$ for lowering and raising the index respectively in cartesian or spherical polar coordinate system given as below. By relation between cartesian and spherical polar coordinate system

$$x = r\sin\theta\cos\varphi , \qquad y = r\sin\theta\sin\varphi ,$$

$$z = r\cos\theta ,$$

so that

$$dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial\theta}d\theta + \frac{\partial x}{\partial\varphi}d\varphi =$$

$$\sin\theta\cos\varphi dr + r\cos\theta\cos\varphi d\theta - r\sin\theta\sin\varphi d\varphi ,$$

$$dy = \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial\theta}d\theta + \frac{\partial y}{\partial\varphi}d\varphi =$$

$$\sin\theta\sin\varphi dr + r\cos\theta\sin\varphi d\theta + r\sin\theta\cos\varphi d\varphi ,$$

$$dz = \frac{\partial z}{\partial r}dr + \frac{\partial z}{\partial\theta}d\theta + \frac{\partial z}{\partial\varphi}d\varphi = \cos\theta dr - r\sin\theta d\theta ,$$

$$\hat{i} = \sin\theta\cos\varphi\,\hat{r} + \cos\theta\cos\varphi\,\hat{\theta} - \sin\varphi\,\hat{\varphi} ,$$

$$\hat{j} = \sin\theta\sin\varphi\,\hat{r} + \cos\theta\sin\varphi\,\hat{\theta} + \cos\varphi\,\hat{\varphi} ,$$

$$\hat{k} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta} ,$$

$$\therefore \mathbf{ds} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k} = dr\,\hat{r} + rd\theta\,\hat{\theta} + r\sin\theta d\varphi\,\hat{\varphi} ,$$

and $ds^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\theta^2$

$$+ r^2\sin^2\theta d\varphi^2 = g_{ik}dx^i dx^k . \tag{6}$$

The components of metric tensor correspondingly in cartesian and spherical polar coordinate system have given as

$$g_{11} = g_{22} = g_{33} = 1,\ g^{11} = g^{22} = g^{33} = 1,$$

$$g_{ij} = g^{ij} = 0, \qquad i \neq j . \tag{7}$$

$$g_{rr} = 1,\ g_{\theta\theta} = r^2,\ g_{\varphi\varphi} = r^2\sin^2\theta ,$$

$$g^{rr} = 1,\ g^{\theta\theta} = 1/r^2 ,$$

$$g^{\varphi\varphi} = 1/r^2\sin^2\theta ,\ g_{\alpha\beta} = g^{\alpha\beta} = 0,\ \alpha \neq \beta . \tag{8}$$

Now we see that $[\mathbf{x} \times \mathbf{p}]_3 = \frac{1}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$,

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta} + \frac{\partial\varphi}{\partial x}\frac{\partial}{\partial\varphi} = \sin\theta\cos\varphi\frac{\partial}{\partial r}$$

$$+ \frac{\cos\theta\cos\varphi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi} ,$$

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial y}\frac{\partial}{\partial\theta} + \frac{\partial\varphi}{\partial y}\frac{\partial}{\partial\varphi} = \sin\theta\sin\varphi\frac{\partial}{\partial r}$$

$$+ \frac{\cos\theta\sin\varphi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi} ,$$

$$\therefore \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial\varphi} ,$$

and $[\mathbf{x} \times \mathbf{p}]_3 = \frac{1}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = \frac{1}{i}\frac{\partial}{\partial\varphi}$.

Since $[\mathbf{x} \times \mathbf{p}]\boldsymbol{\sigma} + \frac{1}{2}$ commutes with $-\rho_1(\boldsymbol{\sigma}.\mathbf{p}) - \rho_3\mu$ and $[\mathbf{x} \times \mathbf{p}]_3 + \frac{1}{2}\sigma_3$ commutes with σ_φ. Therefore third component of $[\mathbf{x}\times\mathbf{p}]$

$+\frac{1}{2}$ $\boldsymbol{\sigma}$ commutes with whole H or to say that $\frac{1}{i}\left(\frac{\partial}{\partial\varphi}\right)+\frac{1}{2}\sigma_3$ commutes with H. We can verify all these steps. Now here we state the important theorem which play central role in obtaining wave equation of required form.

Theorem[5] (pp 23-24): *If in V_n the Hermitian operators and commute, then there exists a complete set of vectors which are simultaneously eigenvectors of A and **B**.*

According to this theorem we can choose ψ in such a way that

$$\left(\frac{1}{i}\frac{\partial}{\partial\varphi}+\frac{1}{2}\sigma_3\right)\psi = \mathrm{M}\psi\,, \tag{9}$$

where M is half an odd integer because eigenvalues of operator $\frac{1}{i}\frac{\partial}{\partial\varphi}$ are integers and eigenvalues of operator $\frac{1}{2}\sigma_3$ are $+\frac{1}{2},-\frac{1}{2}$. Solving above partial differential equation we get

$$\frac{1}{\psi}\partial\psi = i\left(\mathrm{M}-\frac{1}{2}\sigma_3\right)\partial\varphi,$$

$$\psi = \psi' e^{i\left(\mathrm{M}-\frac{1}{2}\sigma_3\right)\varphi}, \tag{10}$$

where ψ' is constant of integration independent of φ.

Now transform σ_1, σ_2, σ_3 as components of a vector to polar coordinates with help of Covariant vector law of tensor calculus[4] as

$$\sigma_r = \frac{\partial x^k}{\partial\xi^r}\sigma_k = \frac{\partial x^k}{\partial r}\sigma_k = \sum_{k=1}^{3}\frac{\partial x^k}{\partial r}\sigma_k = \frac{\partial x^1}{\partial r}\sigma_1 + \frac{\partial x^2}{\partial r}\sigma_2 + \frac{\partial x^3}{\partial r}\sigma_3 = \frac{\partial x}{\partial r}\sigma_1 + \frac{\partial y}{\partial r}\sigma_2 + \frac{\partial z}{\partial r}\sigma_3,$$

$$= \sin\theta\cos\varphi\sigma_1 + \sin\theta\sin\varphi\sigma_2 + \cos\theta\sigma_3$$

$$= \sin\theta\sigma_1\left(\cos\varphi + \sigma_1^{-1}\sigma_2\sin\varphi\right) + \sigma_3\cos\theta\,,$$

$$= \sin\theta\sigma_1\left(\cos\varphi + \sigma_1\sigma_2\sin\varphi\right) + \sigma_3\cos\theta$$

$$= \sin\theta\sigma_1\left(\cos\varphi + i\sigma_3\sin\varphi\right) + \sigma_3\cos\theta\,,$$

$$= \sin\theta\sigma_1 e^{i\sigma_3\varphi} + \sigma_3\cos\theta = \sin\theta\sigma_1 e^{i\sigma_3\varphi/2}e^{i\sigma_3\varphi/2} + e^{-i\sigma_3\varphi/2}e^{i\sigma_3\varphi/2}\sigma_3\cos\theta\,,$$

$$= e^{-i\sigma_3\varphi/2}\sin\theta\sigma_1 e^{i\sigma_3\varphi/2} + e^{-i\sigma_3\varphi/2}e^{i\sigma_3\varphi/2}\sigma_3\cos\theta$$

$$= e^{-i\sigma_3\varphi/2}\left(\sigma_3\cos\theta + \sigma_1\sin\theta\right)e^{i\sigma_3\varphi/2},$$

$$= e^{-i\sigma_3\varphi/2}\sigma_3\left(\cos\theta + \sigma_3^{-1}\sigma_1\sin\theta\right)e^{i\sigma_3\varphi/2}$$

$$= e^{-i\sigma_3\varphi/2}\sigma_3\left(\cos\theta + \sigma_3\sigma_1\sin\theta\right)e^{i\sigma_3\varphi/2},$$

$$= e^{-i\sigma_3\varphi/2}\sigma_3\left(\cos\theta + i\sigma_2\sin\theta\right)e^{i\sigma_3\varphi/2}$$

$$= e^{-i\sigma_3\varphi/2}\sigma_3 e^{i\sigma_2\theta}e^{i\sigma_3\varphi/2}$$

$$= e^{-i\sigma_3\varphi/2}\sigma_3 e^{i\sigma_2\theta/2}e^{i\sigma_2\theta/2}e^{i\sigma_3\varphi/2},$$

$$\sigma_r = e^{-i\sigma_3\varphi/2}e^{-i\sigma_2\theta/2}\sigma_3 e^{i\sigma_2\theta/2}e^{i\sigma_3\varphi/2}\,. \tag{11}$$

$$\sigma_\theta = \frac{\partial x^k}{\partial\xi^\theta}\sigma_k = \frac{\partial x^k}{\partial\theta}\sigma_k = \sum_{k=1}^{3}\frac{\partial x^k}{\partial\theta}\sigma_k = \frac{\partial x^1}{\partial\theta}\sigma_1 + \frac{\partial x^2}{\partial\theta}\sigma_2 + \frac{\partial x^3}{\partial\theta}\sigma_3 = \frac{\partial x}{\partial\theta}\sigma_1 + \frac{\partial y}{\partial\theta}\sigma_2 + \frac{\partial z}{\partial\theta}\sigma_3$$

$$= r\cos\theta\cos\varphi\sigma_1 + r\cos\theta\sin\varphi\sigma_2 - r\sin\theta\sigma_3$$

$$= r\cos\theta\left(\sigma_1\cos\varphi + \sigma_2\sin\varphi\right) - \sigma_3 r\sin\theta\,,$$

$$= r\cos\theta\sigma_1\left(\cos\varphi + \sigma_1^{-1}\sigma_2\sin\varphi\right) - \sigma_3 r\sin\theta$$

$$= r\cos\theta\sigma_1\left(\cos\varphi + \sigma_1\sigma_2\sin\varphi\right) - \sigma_3 r\sin\theta$$

$$= r\cos\theta\sigma_1\left(\cos\varphi + i\sigma_3\sin\varphi\right) - \sigma_3 r\sin\theta$$

$$= r\cos\theta\sigma_1 e^{i\sigma_3\varphi} - \sigma_3 r\sin\theta\,,$$

$$= r\cos\theta\sigma_1 e^{i\sigma_3\varphi/2}e^{i\sigma_3\varphi/2} - e^{-i\sigma_3\varphi/2}e^{i\sigma_3\varphi/2}\sigma_3 r\sin\theta\,,$$

$$= r\cos\theta e^{-i\sigma_3\varphi/2}\sigma_1 e^{i\sigma_3\varphi/2} - e^{-i\sigma_3\varphi/2}\sigma_3 e^{i\sigma_3\varphi/2} r\sin\theta$$
$$= re^{-i\sigma_3\varphi/2}(\sigma_1\cos\theta - \sigma_3\sin\theta)e^{i\sigma_3\varphi/2},$$
$$= re^{-i\sigma_3\varphi/2}\sigma_1(\cos\theta - \sigma_1^{-1}\sigma_3\sin\theta)e^{i\sigma_3\varphi/2}$$
$$= re^{-i\sigma_3\varphi/2}\sigma_1(\cos\theta - \sigma_1\sigma_3\sin\theta)e^{i\sigma_3\varphi/2},$$
$$= re^{-i\sigma_3\varphi/2}\sigma_1(\cos\theta + i\sigma_2\sin\theta)e^{i\sigma_3\varphi/2}$$
$$= re^{-i\sigma_3\varphi/2}\sigma_1 e^{i\sigma_2\theta} e^{i\sigma_3\varphi/2},$$
$$= re^{-i\sigma_3\varphi/2}\sigma_1 e^{i\sigma_2\theta/2} e^{i\sigma_2\theta/2} e^{i\sigma_3\varphi/2}$$
$$= re^{-i\sigma_3\varphi/2} e^{-i\sigma_2\theta/2}\sigma_1 e^{i\sigma_2\theta/2} e^{i\sigma_3\varphi/2},$$
$$\sigma_\theta = re^{-i\sigma_3\varphi/2} e^{-i\sigma_2\theta/2}\sigma_1 e^{i\sigma_2\theta/2} e^{i\sigma_3\varphi/2}. \quad (12)$$

$$\sigma_\varphi = \frac{\partial x^k}{\partial \xi^\varphi}\sigma_k = \frac{\partial x^k}{\partial\varphi}\sigma_k = \sum_{k=1}^{3}\frac{\partial x^k}{\partial\varphi}\sigma_k = \frac{\partial x^1}{\partial\varphi}\sigma_1$$
$$+\frac{\partial x^2}{\partial\varphi}\sigma_2 + \frac{\partial x^3}{\partial\varphi}\sigma_3 = \frac{\partial x}{\partial\varphi}\sigma_1 + \frac{\partial y}{\partial\varphi}\sigma_2 + \frac{\partial z}{\partial\varphi}\sigma_3,$$
$$= -r\sin\theta\sin\varphi\sigma_1 + r\sin\theta\cos\varphi\sigma_2 + 0$$
$$= r\sin\theta(\sigma_2\cos\varphi - \sigma_1\sin\varphi),$$
$$= r\sin\theta\sigma_2(\cos\varphi - \sigma_2^{-1}\sigma_1\sin\varphi)$$
$$= r\sin\theta\sigma_2(\cos\varphi - \sigma_2\sigma_1\sin\varphi),$$
$$= r\sin\theta\sigma_2(\cos\varphi + i\sigma_3\sin\varphi)$$
$$= r\sin\theta\sigma_2 e^{i\sigma_3\varphi} = r\sin\theta\sigma_2 e^{i\sigma_3\varphi/2} e^{i\sigma_3\varphi/2},$$
$$= r\sin\theta e^{-i\sigma_3\varphi/2}\sigma_2 e^{i\sigma_3\varphi/2}$$
$$= r\sin\theta e^{-i\sigma_3\varphi/2}\sigma_2 e^{-i\sigma_2\theta/2} e^{i\sigma_2\theta/2} e^{i\sigma_3\varphi/2},$$
$$= r\sin\theta e^{-i\sigma_3\varphi/2} e^{-i\sigma_2\theta/2}\sigma_2 e^{i\sigma_2\theta/2} e^{i\sigma_3\varphi/2}$$
$$\sigma_\varphi = r\sin\theta e^{-i\sigma_3\varphi/2} e^{-i\sigma_2\theta/2}\sigma_2 e^{i\sigma_2\theta/2} e^{i\sigma_3\varphi/2}. \quad (13)$$

Also $e^{i\sigma_2\theta/2}\dfrac{\partial\psi}{\partial\theta} = \dfrac{\partial}{\partial\theta}\left(e^{i\sigma_2\theta/2}\psi\right) - \dfrac{i\sigma_2}{2}e^{i\sigma_2\theta/2}\psi$

i.e. $e^{i\sigma_2\theta/2}\dfrac{\partial}{\partial\theta} = \left(\dfrac{\partial}{\partial\theta} - \dfrac{i\sigma_2}{2}\right)e^{i\sigma_2\theta/2}$. (14)

Similarly $e^{i\sigma_3\varphi/2}\dfrac{\partial}{\partial\varphi} = \left(\dfrac{\partial}{\partial\varphi} - \dfrac{i\sigma_3}{2}\right)e^{i\sigma_3\varphi/2}$ (15)

and $e^{i\sigma_2\theta/2}e^{i\sigma_3\varphi/2}\dfrac{\partial}{\partial\varphi} = \left(e^{i\sigma_2\theta/2}\dfrac{\partial}{\partial\varphi} - \dfrac{ie^{i\sigma_2\theta/2}\sigma_3}{2}\right)e^{i\sigma_3\varphi/2}$,

$$= \left(e^{i\sigma_2\theta/2}\frac{\partial}{\partial\varphi} - \frac{ie^{i\sigma_2\theta/2}\sigma_3}{2}\right)e^{-i\sigma_2\theta/2}e^{i\sigma_2\theta/2}e^{i\sigma_3\varphi/2},$$

$$= \left(e^{i\sigma_2\theta/2}e^{-i\sigma_2\theta/2}\frac{\partial}{\partial\varphi} - \frac{ie^{i\sigma_2\theta/2}\sigma_3 e^{-i\sigma_2\theta/2}}{2}\right)e^{i\sigma_2\theta/2}e^{i\sigma_3\varphi/2},$$

$$= \left(\frac{\partial}{\partial\varphi} - \frac{ie^{i\sigma_2\theta/2}e^{i\sigma_2\theta/2}\sigma_3}{2}\right)e^{i\sigma_2\theta/2}e^{i\sigma_3\varphi/2} = \left(\frac{\partial}{\partial\varphi} - \frac{ie^{i\sigma_2\theta}\sigma_3}{2}\right)e^{i\sigma_2\theta/2}e^{i\sigma_3\varphi/2},$$

$$= \left(\frac{\partial}{\partial\varphi} - \frac{i}{2}(\cos\theta + i\sigma_2\sin\theta)\sigma_3\right)e^{i\sigma_2\theta/2}e^{i\sigma_3\varphi/2},$$

$$=\left(\frac{\partial}{\partial\varphi}-\frac{i}{2}\left(\sigma_3\cos\theta+i\sigma_2\sigma_3\sin\theta\right)\right)e^{i\sigma_2\theta/2}e^{i\sigma_3\varphi/2},$$

$$\therefore\quad e^{i\sigma_2\theta/2}e^{i\sigma_3\varphi/2}\frac{\partial}{\partial\varphi}=\left(\frac{\partial}{\partial\varphi}-\frac{i}{2}\left(\sigma_3\cos\theta-\sigma_1\sin\theta\right)\right)e^{i\sigma_2\theta/2}e^{i\sigma_3\varphi/2} \tag{16}$$

Now $\sigma^k\frac{\partial}{\partial x^k}=\frac{\partial x^k}{\partial\xi^\alpha}\sigma^\alpha\frac{\partial}{\partial x^k}=\sigma^\alpha\frac{\partial}{\partial\xi^\alpha}=\sum_\alpha\sigma^\alpha\frac{\partial}{\partial\xi^\alpha}=\sigma^r\frac{\partial}{\partial\xi^r}+\sigma^\theta\frac{\partial}{\partial\xi^\theta}+\sigma^\varphi\frac{\partial}{\partial\xi^\varphi}$,

$$=\sigma_r g^{rr}\frac{\partial}{\partial r}+\sigma_\theta g^{\theta\theta}\frac{\partial}{\partial\theta}\sigma_\varphi g^{\varphi\varphi}\frac{\partial}{\partial\varphi}=\sigma_r\frac{1}{1}\frac{\partial}{\partial r}+\sigma_\theta\frac{1}{r^2}\frac{\partial}{\partial\theta}\sigma_\varphi\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\varphi}, \tag{17}$$

Put the values in equation (17) we get

$$\sigma^k\frac{\partial}{\partial x^k}=e^{-i\sigma_2\theta/2}e^{-i\sigma_3\varphi/2}\left[\sigma_3 e^{i\sigma_2\theta/2}e^{i\sigma_3\varphi/2}\frac{\partial}{\partial r}+\frac{1}{r}\sigma_1 e^{i\sigma_2\theta/2}e^{i\sigma_3\varphi/2}\frac{\partial}{\partial\theta}+\frac{1}{r\sin\theta}\sigma_2 e^{i\sigma_2\theta/2}e^{i\sigma_3\varphi/2}\frac{\partial}{\partial\varphi}\right]$$

$$=e^{-i\sigma_2\theta/2}e^{-i\sigma_3\varphi/2}\left[\sigma_3 e^{i\sigma_2\theta/2}e^{i\sigma_3\varphi/2}\frac{\partial}{\partial r}+\frac{1}{r}\sigma_1\left(\frac{\partial}{\partial\theta}-\frac{i\sigma_2}{2}\right)e^{i\sigma_2\theta/2}e^{i\sigma_3\varphi/2}\right.$$

$$\left.+\frac{1}{r\sin\theta}\sigma_2 e^{i\sigma_2\theta/2}\left(\frac{\partial}{\partial\varphi}-\frac{i\sigma_3}{2}\right)e^{i\sigma_3\varphi/2}\right],$$

$$=e^{-i\sigma_2\theta/2}e^{-i\sigma_3\varphi/2}\left[\sigma_3 e^{i\sigma_2\theta/2}e^{i\sigma_3\varphi/2}\frac{\partial}{\partial r}+\frac{1}{r}\sigma_1\left(\frac{\partial}{\partial\theta}-\frac{i\sigma_2}{2}\right)e^{i\sigma_2\theta/2}e^{i\sigma_3\varphi/2}\right.$$

$$\left.+\frac{1}{r\sin\theta}\sigma_2\left\{\frac{\partial}{\partial\varphi}-\frac{i}{2}\left(\sigma_3\cos\theta-\sigma_1\sin\theta\right)\right\}e^{i\sigma_2\theta/2}e^{i\sigma_3\varphi/2}\right],$$

$$\sigma^k\frac{\partial}{\partial x^k}=e^{-i\sigma_2\theta/2}e^{-i\sigma_3\varphi/2}\left[\sigma_3\frac{\partial}{\partial r}+\frac{1}{r}\sigma_1\left(\frac{\partial}{\partial\theta}-\frac{i\sigma_2}{2}\right)\right.$$

$$\left.+\frac{1}{r\sin\theta}\sigma_2\left\{\frac{\partial}{\partial\varphi}-\frac{i}{2}\left(\sigma_3\cos\theta-\sigma_1\sin\theta\right)\right\}\right]e^{i\sigma_2\theta/2}e^{i\sigma_3\varphi/2}. \tag{18}$$

For obtaining equations (11) to (18) we use following properties of Pauli operators $(\sigma_1,\sigma_2,\sigma_3)$:

i. $\sigma_1\sigma_2\sigma_3=i$, $\sigma_1^2=\sigma_2^2=\sigma_3^2=1$, i.e. $\sigma_1^{-1}=\sigma_1$, $\sigma_2^{-1}=\sigma_2$, $\sigma_3^{-1}=\sigma_3$,

$\sigma_1\sigma_2 = -\sigma_2\sigma_1$, $\sigma_1\sigma_3 = -\sigma_3\sigma_1$, $\sigma_2\sigma_3 = -\sigma_3\sigma_2$.

ii. $\sigma_3 e^{-i\sigma_3\varphi/2} = e^{-i\sigma_3\varphi/2}\sigma_3$, *i.e.* if a linear operator A commutes with B then A also commutes with function of B.

iii. $\sigma_1 e^{-i\sigma_3\varphi/2} = e^{i\sigma_3\varphi/2}\sigma_1$, $e^{i\sigma_2\theta} = \cos\theta + i\sigma_2\sin\theta$.

4. Required Schrödinger equation in polar form

Hence on putting (10) in (2) we get wave equation suitable for problem

$$\{ -\rho_1(\boldsymbol{\sigma}.\mathbf{p}) - \rho_3\mu - \rho_1\frac{\sigma_\varphi}{r^2\sin^2\theta}\frac{n}{2}(1-\cos\theta)\}\psi = E\psi ,$$

$$\left\{\frac{\rho_1}{i}\left(\sigma_1\frac{\partial}{\partial x} + \sigma_2\frac{\partial}{\partial y} + \sigma_3\frac{\partial}{\partial z}\right) + \frac{\rho_1\sigma_\varphi}{r^2\sin^2\theta}\frac{n}{2}(1-\cos\theta) + \rho_3\mu\right\}\psi + E\psi = 0 ,$$

$$\left\{\frac{\rho_1}{i}\left(\sigma^1 g_{11}\frac{\partial}{\partial x^1} + \sigma^2 g_{22}\frac{\partial}{\partial x^2} + \sigma^3 g_{33}\frac{\partial}{\partial x^3}\right) + \frac{\rho_1\sigma_\varphi}{r^2\sin^2\theta}\frac{n}{2}(1-\cos\theta) + \rho_3\mu\right\}\psi + E\psi = 0 ,$$

$$\frac{\rho_1}{i}\left\{\sigma^k\frac{\partial}{\partial x^k} + \frac{i\sigma_\varphi}{r^2\sin^2\theta}\frac{n}{2}(1-\cos\theta)\right\}\psi + \{\rho_3\mu + E\}\psi = 0 ,$$

$$\frac{\rho_1}{i}e^{-i\sigma_2\theta/2}e^{-i\sigma_3\varphi/2}\left\{\sigma_3\frac{\partial}{\partial r} + \frac{\sigma_1}{r}\left(\frac{\partial}{\partial\theta} - \frac{i\sigma_2}{2}\right) + \frac{\sigma_2}{r\sin\theta}\left[\frac{\partial}{\partial\varphi} - \frac{i}{2}(\sigma_3\cos\theta - \sigma_1\sin\theta)\right]\right.$$

$$\left. + \frac{ir\sin\theta\sigma_2}{r^2\sin^2\theta}\frac{n}{2}(1-\cos\theta)\right\}e^{i\sigma_2\theta/2}e^{i\sigma_3\varphi/2}\psi + \{\rho_3\mu + E\}\psi = 0 ,$$

$$\frac{\rho_1}{i}e^{-i\sigma_2\theta/2}e^{-i\sigma_3\varphi/2}\left\{\sigma_3\frac{\partial}{\partial r} + \frac{\sigma_1}{r}\frac{\partial}{\partial\theta} - \frac{i\sigma_1\sigma_2}{2r} + \frac{\sigma_2}{r\sin\theta}\frac{\partial}{\partial\varphi} - \frac{i\sigma_2\sigma_3\cos\theta}{2r\sin\theta} + \frac{i\sigma_2\sigma_1\sin\theta}{2r\sin\theta}\right.$$

$$\left. + \frac{i\sigma_2}{r\sin\theta}\frac{n}{2}(1-\cos\theta)\right\}e^{i\sigma_2\theta/2}e^{i\sigma_3\varphi/2}e^{i\left(M-\frac{1}{2}\sigma_3\right)\varphi}\psi' + \{\rho_3\mu + E\}e^{i\left(M-\frac{1}{2}\sigma_3\right)\varphi}\psi' = 0 ,$$

pre multiply $e^{i\sigma_2\theta/2}$ in both side of above equation and then set $e^{i\sigma_2\theta/2}\psi' = \psi_0$ we get

$$\frac{\rho_1}{i}e^{-i\sigma_3\varphi/2}\left\{\sigma_3\frac{\partial}{\partial r} + \frac{\sigma_1}{r}\frac{\partial}{\partial\theta} - \frac{i\sigma_1\sigma_2}{2r} + \frac{\sigma_2}{r\sin\theta}iM - \frac{i\sigma_2\sigma_3\cos\theta}{2r\sin\theta} + \frac{i\sigma_2\sigma_1}{2r}\right.$$

$$+\frac{i\sigma_2}{r\sin\theta}\frac{n}{2}(1-\cos\theta)\Big\}e^{i\sigma_3\varphi/2}e^{i\left(M-\frac{1}{2}\sigma_3\right)\varphi}\psi_0+\{\rho_3\mu+E\}e^{i\left(M-\frac{1}{2}\sigma_3\right)\varphi}\psi_0=0,$$

$$\frac{\rho_1}{i}e^{-i\sigma_3\varphi/2}\Big\{\sigma_3\frac{\partial}{\partial r}+\frac{\sigma_1}{r}\frac{\partial}{\partial\theta}+\frac{\sigma_3}{2r}-\frac{\sigma_1\sigma_3}{r\sin\theta}M+\frac{\sigma_1\cos\theta}{2r\sin\theta}+\frac{\sigma_3}{2r}$$

$$-\frac{\sigma_1\sigma_3}{r\sin\theta}\frac{n}{2}(1-\cos\theta)\Big\}e^{i\sigma_3\varphi/2}e^{i\left(M-\frac{1}{2}\sigma_3\right)\varphi}\psi_0+\{\rho_3\mu+E\}e^{i\left(M-\frac{1}{2}\sigma_3\right)\varphi}\psi_0=0,$$

pre multiply $e^{i\sigma_3\varphi/2}$ and then post multiply $e^{-i\sigma_3\varphi/2}e^{-i\left(M-\frac{1}{2}\sigma_3\right)\varphi}$ from above equation we get

$$\frac{\rho_1}{i}\left\{\sigma_3\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)+\frac{\sigma_1}{r}\left(\frac{\partial}{\partial\theta}-\frac{\sigma_3}{\sin\theta}M-\frac{\sigma_3}{\sin\theta}\frac{n}{2}(1-\cos\theta)+\frac{\cos\theta}{2\sin\theta}\right)\right\}\psi_0+\{\rho_3\mu+E\}\psi_0=0,$$

$$\frac{\rho_1}{i}\left\{\sigma_3\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)+\frac{\sigma_1}{r}\left(\frac{\partial}{\partial\theta}-\frac{\sigma_3}{\sin\theta}M-\frac{\sigma_3}{\sin\theta}\frac{n}{2}(1-\cos\theta)+\frac{{\sigma_3}^2\cos\theta}{2\sin\theta}\right)\right\}\psi_0+\{\rho_3\mu+E\}\psi_0=0$$

$$\left[\frac{1}{i}\rho_1\left\{\sigma_3\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)+\frac{\sigma_1}{r}\left(\frac{\partial}{\partial\theta}-\frac{\sigma_3}{\sin\theta}\left\{M+\frac{n}{2}(1-\cos\theta)-\frac{\sigma_3}{2}\cos\theta\right\}\right)\right\}+\rho_3\mu+E\right]\psi_0=0.$$

This is required form of wave equation of given problem. This is partial differential equation in $\partial/\partial r$, $\partial/\partial\theta$ and wave function ψ_0 is function of only (r, θ). To remove the differential term $\partial/\partial\varphi$ we have used indirect method of separation of variables for solving the partial differential equation.

5. Conclusion

The above calculation helps in study of the whole paper[1] and gives a new way to solve a partial differential equation.

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References

1. Harish-Chandra, Motion of an Electron in the Field of a Magnetic Pole, *Physical Review*, Vol. *74*, No. 8, pp-883-887 (1948).
2. P. A. M. Dirac, The Principles of Quantum Mechanics, Clarendon Press, Oxford (1958).
3. P.A.M. Dirac, The Quantum Theory of the Electron, Proceedings of the Royal Society, Series A, Vol.117, No.778, pp.610-624 (1928).
4. C. E. Weatherburn, An Introduction to Riemannian Geometry and the Tensor Calculus, Cambridge University Press (1966).
5. R.A. Newing and J. Cunningham, Quantum Mechanics, Oliver & Boyd Ltd, Interscience Publishers Inc., A division of John Wiley & Sons, Inc., (1967).