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On some generating functions of modified jacobi polynomials

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Abstract

In this note, we have obtained a novel extension of a theorem on bilateral generating function of modified Jacobi polynomials from the existence of quasi - bilateral generating function by group theoretic method.

Key words : generating functions, Jacobi polynomials.

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1. Introduction

In[1], the quasi bilateral (/ bilinear) generating function is defined by

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n \; p_n^{(\alpha)}(x)\, q_m^{(n)}(u) \; w^n \tag{1.1}$$

where a_n, the coefficients are quite arbitrary and $p_n^{(\alpha)}(x)$, $q_m^{(n)}(u)$ are two special functions of orders n, m and of parameters of α and n respectively. If $q_m^{(n)}(u) \equiv p_m^{(n)}(u)$, the generating relation is known as quasi bilinear.

In[2], Alam and Chongdar have proved the following theorem on bilateral generating function involving $P_n^{(\alpha+n,\beta)}(x)$, a modification of Jacobi polynomial by group- theoretic method.

Theorem 1 : If there exists a unilateral generating function

$$G(x,w) = \sum_{n=0}^{\infty} a_n P_n^{(\alpha+n,\beta)}(x)\, w^n, \tag{1.2}$$

then

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$$(1+w)^{\alpha}\left[1+\frac{w}{2}(1-x)\right]^{-1-\alpha-\beta} G\left(\frac{x-\frac{w}{2}(1-x)}{1+\frac{w}{2}(1-x)}, \frac{wv\,(1+w)}{\{1+\frac{w}{2}(1-x)\}^{2}}\right)$$

$$=\sum_{n=0}^{\infty} w^{n}\ \sigma_{n}(v,x)$$

where

$$\sigma_{n}(v,x)=\sum_{p=0}^{n} a_{p}\binom{n}{p} P_{n}^{(\alpha-n+2p,\,\beta)}(x)\ v^{p}\ . \tag{1.4}$$

The importance of the above theorem lies in the fact that whenever a unilateral generating relation of the form (1.2) is known then the corresponding bilateral generating relation can at once be written down from (1.3). Thus a large number of bilateral generating relations can be obtained by attributing different values to a_n in (1.2).

The aim at writing this note is to show that the existence of a quasi- bilinear generating function implies the existence of a more general generating function from the group theoretic view point.

In the present paper, we have obtained the following extension of the above theorem from the existence of quasi bilateral generating relation as defined in (1.1).

Theorem 2: If there exists a generating relation

$$G(x,u,w)=\sum_{n=0}^{\infty} a_{n} P_{n}^{(\alpha+n,\,\beta)}(x)\, P_{m}^{(n,\beta)}(u)\, w^{n}, \tag{1.5}$$

then

$$(1.6)\qquad (1-w)^{-1-\beta-m}\left[1-(1-x)w\right]^{-1-\alpha-\beta}(1-2w)^{\alpha}$$

$$\times\ G\left(\frac{x+(1-x)w}{1-(1-x)w)}, \frac{u+w}{1-w}, \frac{wv}{1-w}\frac{(1-2w)}{\{1-(1-x)w\}^{2}}\right)$$

$$=\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_{n}\frac{w^{n+p+q}}{p!\,q!}\ (-2)_{p}(n+1)_{p}(1+n+\beta+m)_{q}\ P_{n+p}^{(\alpha+n-p,\,\beta)}(x)\, P_{m}^{(n+q,\,\beta)}(u)\, v^{n}.$$

2 Proof of the theorem :

For the modified Jacobi polynomial, we first consider the following operators[2,3]

$$R_1 = (1-x^2)y^{-2}z\ \frac{\partial}{\partial x}\ .-(x+1)y^{-1}z\ \frac{\partial}{\partial y}$$

$$-2xy^{-2}z^2\ \frac{\partial}{\partial z} - (1+\beta)(x-1)\ y^{-2}z\quad .$$

and

$$R_2 = (1+u)\ t\ \frac{\partial}{\partial u} + t^2\ \frac{\partial}{\partial t} + (1+\beta+m)\ t.$$

such that

$$R_1(P_n^{(\alpha+n,\,\beta)}(x)y^{\alpha}z^n) = -2(n+1)$$

$$P_{n+1}^{(\alpha+n-1,\,\beta)}(x)\ y^{\alpha-2}z^{n-1}\ . \qquad (2.1)$$

and

$$R_2\left(P_m^{(n,\beta)}(u)\ t^n\right) = (1+n+\beta+m)\ P_m^{(n+1,\beta)}\ (u)\ t^{n+1}\ . \qquad (2.2)$$

The extended form of the groups corresponding to R_1 and R_2 are given by

$$e^{wR_1}\ f(x,y,z) = \left[1-(1-x)y^{-2}zw\right]^{-\beta-1}$$

$$\times f\left(\frac{x+(1-x)y^{-2}zw}{1-(1-x)y^{-2}zw}, \frac{y\,(1-2y^{-2}zw)}{1-(1-x)y^{-2}zw},\right.$$

$$\left.\frac{z\,(1-2y^{-2}zw)}{\{1-(1-x)y^{-2}zw\}^2}\right)$$

and

$$e^{w\,R_2}\ f(u,t) = (1-wt)^{-1-\beta-m}\quad f\left(\frac{u+wt}{1-wt}, \frac{t}{1-wt}\right). \qquad (2.4)$$

We now consider the following formula

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n\ P_n^{(\alpha+n,\,\beta)}(u)\ P_m^{(n,\beta)}(u)\ w^n. \qquad (2.5)$$

Replacing w by $wztv$ in (2.5), we get

$$G(x,u,wztv) = \sum_{n=0}^{\infty} a_n\ P_n^{(\alpha+n,\,\beta)}(u)\ P_m^{(n,\beta)}(u)\ (wztv)^n. \qquad (2.6)$$

Now multiplying both sides of (2.6) by y^{α} we get

$$y^{\alpha}G(x,u,wztv) = \sum_{n=0}^{\infty} a_n\left(P_n^{(\alpha+n\,,\,\beta)}(x)y^{\alpha}z^n\right)$$

$$\left(P_m^{(n\,,\beta)}(u)t^n\right)\ (wv)^n \qquad (2.7)$$

Now operating $e^{wR_1}e^{wR_2}$ on both sides of (2.7) we get,

$$e^{w\,R_1}e^{wR_2}\left(y^{\alpha}G(x,u,wztv)\right)$$

$$= e^{w\,R_1}e^{w\,R_2}\left(\sum_{n=0}^{\infty} a_n\left(P_n^{(\alpha+n\,,\beta)}(x)y^{\alpha}z^n\right)\right.$$

$$\left.\left(P_m^{(n\,,\beta)}(u)t^n\right)(wv)^n\right). \qquad (2.8)$$

Left hand sides of (2.8)

$$(2.9)\qquad = e^{w\,R_1}e^{w\,R_2}\left(y^{\alpha}\ G(x,u,wztv)\right)$$

$$= e^{w\,R_1}\left((1-wt)^{-1-\beta-m}\ y^{\alpha}\ G(\ x, \frac{u+wt}{1-wt}, \frac{wztv}{1-wt}\ )\right)$$

$$= (1-wt)^{-1-\beta-m}\left[1-(1-x)y^{-2}zw\right]^{-1-\alpha-\beta} y^{\alpha}\ (1-2\,y^{-2}zw)^{\alpha}$$

$$\times G\left(\frac{x+(1-x)y^{-2}zw}{1-(1-x)y^{-2}zw)}, \frac{u+wt}{1-wt}, \frac{wztv}{1-wt}\ \frac{(1-2y^{-2}zw)}{\{1-(1-x)y^{-2}zw\}^{2}}\right)$$

Right hand sides of (2.8)

$$(2.10) \qquad = \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_n \frac{w^{n+p+q}}{p!\ q!} v^n\ (-2)_p (n+1)_p (1+n+\beta+m)_q$$

$$\times P_{n+p}^{(\alpha+n-p,\,\beta)}(x)\ y^{\alpha-2p}\ z^{n+p}\ P_m^{(n+q,\,\beta)}(u)\ t^{n+q}$$

Equating (2.9) and (2.10) and then putting $y = z = t = 1,$ we get

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$$(1-w)^{-1-\beta-m}\ [1-(1-x)w]^{-1-\alpha-\beta}\ (1-2w)^{\alpha}$$

$$\times G\left(\frac{x+(1-x)w}{1-(1-x)w}, \frac{u+w}{1-w}, \frac{wv}{1-w}\ \frac{(1-2w)}{\{1-(1-x)w\}^{2}}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_n \frac{w^{n+p+q}}{p!\,q!}\ (-2)_p (n+1)_p (1+n+\beta+m)_q\ P_{n+p}^{(\alpha+n-p,\,\beta)}(x)\ P_m^{(n+q,\,\beta)}(u)\, v^n,$$

which is theorem 2 .

Corollary : Putting $m = 0$, we get

$$(1-w)^{-1-\beta}\ [1-(1-x)w]^{-1-\alpha-\beta}\ (1-2w)^{\alpha}$$

$$\times G\left(\frac{x+(1-x)w}{1-(1-x)w}, \frac{wv}{1-w}\ \frac{(1-2w)}{\{1-(1-x)w\}^{2}}\right).$$

$$= \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} a_n \frac{w^{n+p}}{p!} v^n (-2)^{n+p} \frac{1}{(-2)^n} (n+1)_p P_{n+p}^{(\alpha+n-p,\beta)}(x) \sum_{q=0}^{\infty} w^q \frac{(1+n+\beta)_q}{q!}$$

$$= \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} a_n \frac{(-2w)^{n+p}}{p!} \left(-\frac{v}{2}\right)^n (n+1)_p P_{n+p}^{(\alpha+n-p,\beta)}(x) (1-w)^{-1-n-\beta} \quad .$$

$$= \sum_{n=0}^{\infty} (-2w)^{n+p} \sum_{p=0}^{\infty} a_n \frac{(n+1)_p}{p!} P_{n+p}^{(\alpha+n-p,\beta)}(x) \left(-\frac{v}{2(1-w)}\right)^n (1-w)^{-1-\beta}$$

$$= (1-w)^{-1-\beta} \sum_{n=0}^{\infty} (-2w)^n \sum_{p=0}^{n} a_{n-p} \frac{(n-p+1)_p}{p!} P_n^{(\alpha+n-2p,\beta)}(x) \left(-\frac{v}{2(1-w)}\right)^{n-p} .$$

Then we have

$$\left[1-(1-x)w\right]^{-1-\alpha-\beta} (1-2w)^{\alpha} G\left(\frac{x+(1-x)w}{1-(1-x)w}, \frac{wv}{1-w} \frac{(1-2w)}{\{1-(1-x)w\}^2} \right)$$

$$= \sum_{n=0}^{\infty} (-2w)^n \sum_{p=0}^{n} a_p \frac{(p+1)_{n-p}}{(n-p)!} P_n^{(\alpha-n+2p,\beta)}(x) \left(-\frac{v}{2(1-w)}\right)^p .$$

Replacing $-2w$ by w and $-\dfrac{v}{2(1-w)}$ by v, we get

$$(1+w)^{\alpha} [1+\frac{w}{2}(1-x)]^{-1-\alpha-\beta} G \left(\frac{x-\frac{w}{2}(1-x)}{1+\frac{w}{2}(1-x)}, \frac{wv(1+w)}{\{1+\frac{w}{2}(1-x)\}^2} \right)$$

$$= \sum_{n=0}^{\infty} w^n \sigma_n(v,x)$$

where

$$\sigma_n(v,x) = \sum_{p=0}^{n} a_p \binom{n}{p} P_n^{(\alpha-n+2p,\beta)}(x) \; v^p ,$$

which is the theorem 1.

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