Ultra Scientist Vol. 28(1)A, 65-70 (2016). ISSN 2231-3478(Print) ISSN 2319-8052 (Online)

Baire Category Theorem in terms of weak open sets in Bitopological Spaces

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(Acceptance Date 25th April, 2016)

Abstract

This paper introduce and establish the properties of (1,2)*α_ψ-compact sets, (1,2)*α_ψ-locally compact sets, (1,2)*α_ψ-Hausdorff spaces. Also we developed a main theorem called Baire Category Theorem by using weak open sets in bitopological spaces.

Key words: Bitopological spaces, (1,2)*α_ψ -open sets, (1,2)*α_ψ-dense sets, (1,2)*α_ψ-Baire spaces, (1,2)*α_ψ-compact set and (1,2)*α_ψ –Hausdorff spaces.

MSC 2010: 54A05, 54A52, 54D10

1. Introduction

The bitopological space concept was first introduced by Kelly[4] in 1963, which is a non-empty set X equipped with two arbitrary topologies τ_1 and τ_2 defined on X. In this spaces, Fukutake[1] defined the Baire space concepts in 1992. Lellis Thivagar *et al.*[5,8] introduced (1, 2)*α-open sets in bitopological spaces by defining a new class of open sets namely $\tau_{1,2}$-open sets and in it we can observe that the family of (1, 2)*α-open sets is need not form a topology but an m-structure. Also Lellis Thivagar *et al.*[6] studied and characterized the (1, 2)*α_ψ-Baire space by using (1, 2)*α_ψ-open sets and introduced (1, 2)*α_ψ-nowhere dense sets, (1, 2)*α_ψ-dense sets etc. This paper introduce (1, 2)*α_ψ-compact set, (1, 2)*α_ψ -Hausdorff spaces and also focus on to develop the main theorem called Baire Category Theorem[2-3].

2. Preliminaries :

In this preliminary section, we discuss

briefly a number of basic definitions and some existing results which are useful in the sequel. Throughout this paper, (X, τ_1, τ_2) (simply X) represents bitopological spaces on which no separation axioms are assumed, unless otherwise mentioned.

Definition 2.1.[4] A non-empty set X together with two arbitrary topologies τ_1 and τ_2 is called a bitopological space and is denoted by (X, τ_1, τ_2).

Definition 2.2.[5] A subset S of a bitopological space (X, τ_1, τ_2) is called $\tau_{1,2}$-open if and only if $S=A\cup B$, where A is τ_1-open and B is τ_2-open.The complement of $\tau_{1,2}$-open sets are called $\tau_{1,2}$-closed sets.

Remark 2.3.[5] Let A be a subset of a bitopological space (X, τ_1, τ_2). Then

i. $\tau_{1,2}$-int(A) $=\cup\{G: G\subseteq A$ and G is $\tau_{1,2}$-open$\}$

ii. $\tau_{1,2}$-cl(A) $=\cap\{F: A\subseteq F$ and F is $\tau_{1,2}$-closed$\}$.

Definition 2.4. A subset A of a bitopological space (X, τ_1, τ_2) is called $(1,2)^*\alpha$-open[5] if $A\subseteq\tau_{1,2}\text{-int}(\tau_{1,2}\text{-cl}(\tau_{1,2}\text{-int}(A)))$ and the complement of $(1,2)^*\alpha$-open sets are called $(1,2)^*\alpha$-closed sets. The family of all $(1,2)^*\alpha$-open setsneed not form a topology and is denoted by $(1,2)^*\alpha O(X)$.

Definition 2.5.[6] A topology which is generated by the family $(1,2)^*\alpha O(X)$ as its subbasis and the collection of elements of this topology is denoted by $(1,2)^*\alpha_\psi O(X)$. A subset A of X is called $(1,2)^*\alpha_\psi$-open if $A\in(1,2)^*\alpha_\psi O(X)$. From this it is very clear that every $(1,2)^*\alpha$-open set is $(1,2)^*\alpha_\psi$-open butnot converse and the complement of a $(1,2)^*\alpha_\psi$-open set is $(1,2)^*\alpha_\psi$-closedset. The collection of all $(1,2)^*\alpha_\psi$-closed sets is denoted by $(1,2)^*\alpha_\psi C(X)$.

Remark 2.6.[6] Let A be a subset of a bitopological space X. Then $(1,2)^*\alpha_\psi$-interior and $(1,2)^*\alpha_\psi$-closure of A are defined as follows:

i) $(1,2)^*\alpha_\psi$-int(A) $=\cup\{G: G\subseteq A$ and G is $(1,2)^*\alpha_\psi$-open$\}$

ii) $(1,2)^*\alpha_\psi$-cl(A) $=\cap\{F: A\subseteq F$ and F is $(1,2)^*\alpha_\psi$-closed$\}$.

Definition 2.7.[6] A subset A of a bitopological space X is called

i) $(1,2)^*\alpha_\psi$-dense if $(1,2)^*\alpha_\psi$-cl(A) $=X$

ii) $(1,2)^*\alpha_\psi$-nowhere dense if $(1,2)^*\alpha_\psi$-int$((1,2)^*\alpha_\psi$-cl(A)$) = \phi$.

Theorem 2.8.[6] A subset A of X is $(1,2)^*\alpha_\psi$-nowhere dense if and only if $X-(1,2)^*\alpha_\psi$-cl(A) is $(1,2)^*\alpha_\psi$-dense in X.

Definition 2.9.[6] A bitopological spaceXis said to be $(1,2)^*\alpha_\psi$-Baire spaceif for any countable collection $\{A_n\}$of $(1,2)^*\alpha_\psi$-closed subsets of X such that$(1,2)^*\alpha_\psi$-int(A_n) $=\phi\,\forall n$, then $(1,2)^*\alpha_\psi$-int$(\cup_n A_n) = \phi$.

Theorem 2.10.[6] A bitopological space X is $(1,2)^*\alpha_\psi$-Baire space if and onlyif for any countable collection $\{A_n\}$of $(1,2)^*\alpha_\psi$-open subsets of X such that$(1,2)^*\alpha_\psi$-cl(A_n) $=X\ \forall n$, then $(1,2)^*\alpha_\psi$-cl$(\cap_n A_n)$ $=X$.

Definition 2.11.[6] A subset A of a

bitopological spaceXis said to be $(1,2)^*\alpha_\psi$-first category if $A=\cup_n A_n$, where each A_n is $(1,2)^*\alpha_\psi$-nowhere dense subsetof X. If A is not $(1,2)^*\alpha_\psi$-first category, then A is said to be $(1,2)^*\alpha_\psi$-secondcategory.

Proposition 2.12. [6] If A is a $(1,2)^*\alpha_\psi$-first category subset of a $(1,2)^*\alpha_\psi$-Baire space X, then $(1,2)^*\alpha_\psi\text{-int}(A) = \phi$.

Proposition 2.13.[6] Any $(1,2)^*\alpha_\psi$-open subspace Y of a $(1,2)^*\alpha_\psi$-Baire spaceX is itself a $(1,2)^*\alpha_\psi$-Baire Space.

3. Baire Category Theorem :

In this section we define and characterize the properties of $(1,2)^*\alpha_\psi$ -compact, $(1,2)^*\alpha_\psi$-locally compact and $(1,2)^*\alpha_\psi$-Hausdorff space. Usingthese concepts, we derived Baire Category Theorem.

Definition 3.1 Let Λ be an index set, $\{O_i\}_{i\in\Lambda}$be a family of $(1,2)^*\alpha_\psi$ -open subsets of a bitopological space X. Let A be a subset of X. Then $\{O_i\}_{i\in\Lambda}$ is said to be $(1,2)^*\alpha_\psi$ -open cover of A if $A \subseteq \cup_{i\in\Lambda} O_i$. A finite subfamily $\{O_{i1}, O_{i2}, ..., O_{in}\}$ of $\{O_i\}_{i\in\Lambda}$ is said to be finite sub cover of A if $A \subseteq O_{i1} \cup O_{i2} \cup ... \cup O_{in}$.

Definition 3.2. A subset A of a bitopological space X is said to be $(1,2)^*\alpha_\psi$-compact if every $(1,2)^*\alpha_\psi$-open cover of A has a finite $(1,2)^*\alpha_\psi$-sub cover.

Definition 3.3. A bitopological space X is said to be $(1,2)^*\alpha_\psi$-locally compact when every point of X has an $(1,2)^*\alpha_\psi$-open neighborhood with $(1,2)^*\alpha_\psi$-compact closure.

Remark 3.4. Every $(1,2)^*\alpha_\psi$-compact space is $(1,2)^*\alpha_\psi$-locally compact and every finite bitopological space is $(1,2)^*\alpha_\psi$-compact.

Definition 3.5. A bitopological space X is said to be $(1,2)^*\alpha_\psi$-Hausdorff space if every pair of distinct points a, b $\in$ X there exists $(1,2)^*\alpha_\psi$-open sets A and B such that a $\in$A, b $\in$B and $A\cap B=\phi$.

Example 3.6. Let X= {a, b, c, d}, τ_1= {ϕ, X, {a}, {b}, {a, b}}, τ_2 = {ϕ, X, {c}, {d}, {c, d}}. Then $\tau_{1,2}O(X) = P(X) = (1,2)^*\alpha O(X) = (1,2)^*\alpha_\psi O(X)$. Here X is $(1,2)^*\alpha_\psi$-Hausdorff space.

Lemma 3.7. Let A be a $(1,2)^*\alpha_\psi$-compact subset of $(1,2)^*\alpha_\psi$-Hausdorff space and suppose that $p \in X{-}A$. Then there are $(1,2)^*\alpha_\psi$-open sets G and H such that $A\subseteq G$, $p\in H$ and $G\cap H=\phi$.

Proof: Consider an element $a \in A$. Since X is $(1,2)^*\alpha_\psi$-Hausdorff space, there are $(1,2)^*\alpha_\psi$-open sets $G_a\subseteq X$ and $H_a\subseteq X$, such that $a \in G_a$, $p \in H_a$ and $G_a \cap H_a = \phi$. Note that the collection $\{G_a : a \in A\}$ is an $(1,2)^*\alpha_\psi$-open cover of A. Since A is $(1,2)^*\alpha_\psi$-compact, there is a finite collection $a_1, a_2, ..., a_n$ such that $A\subseteq G_{a1}\cup G_{a2}\cup ... \cup G_{an}$. Set $G=G_{a1}\cup G_{a2}\cup ... \cup G_{an}$ and $H = H_{a1} \cap H_{a2} \cap ... \cap H_{an}$. Furthermore, $A \subseteq G$ and $p \in H$ since $p \in H_{ai}$ each i. We claim that $G \cap H = \phi$. Note that each $G_{ai} \cap$

$H_{ai} = \phi$ implies that $G_{ai} \cap H = \phi$ and then by the distributive law, $(G_{a1} \cup G_{a2} \cup \ldots \cup G_{an}) \cap H = (G_{a1} \cap H) \cup (G_{a2} \cap H) \cup \ldots \cup (G_{an} \cap H) = \phi$.

Lemma 3.8. Let A be a $(1,2)^*\alpha_\psi$-compact subset of $(1,2)^*\alpha_\psi$-Hausdorff space and suppose that $p \notin A$. Then there is a $(1,2)^*\alpha_\psi$-open set G such that $p \in G \subseteq X - A$.

Proof: By above Lemma, there are $(1,2)^*\alpha_\psi$-open sets $G \subseteq X$ and $H \subseteq X$, such that $p \in G$, $A \subseteq H$ and $G \cap H = \phi$. Hence $G \cap A = \phi$, and $p \in G \subseteq X$-A.

Corollary 3.9. In a $(1,2)^*\alpha_\psi$-Hausdorff space, any $(1,2)^*\alpha_\psi$- compact subset is $(1,2)^*\alpha_\psi$-closed.

Proof: Let A be a $(1,2)^*\alpha_\psi$- compact subset of a $(1,2)^*\alpha_\psi$-Hausdorff space X. We equivalently prove that X A is $(1,2)^*\alpha_\psi$-open. Let $p \in X$-A, $p \notin A$. Then by above Lemma, there is a $(1,2)^*\alpha_\psi$-open set G_p such that $p \in G_p \subseteq X$-A. Hence X $A = \cup G_p$ which is $(1,2)^*\alpha_\psi$-open.

Lemma 3.10. Let $\{A_n: n \in \Lambda\}$ be a collection of $(1,2)^*\alpha_\psi$-compact subset of $(1,2)^*\alpha_\psi$-Hausdorff space X. If $\cap_{n \in \Lambda} A_n = \phi$, there is a finite subset $F \subset \Lambda$ such that $\cap_{n \in F} A_n = \phi$.

Proof: Fix one of the A_n's; say A_{n1}. Since $\cap_{n \in \Lambda} A_n = \phi$, every elementof$A_{n1}$ is in the complement X-A_n of A_n, for some n. This means that $\{X\text{-}A_n: n \in \Lambda\}$ covers A_{n1}. By above corollary, each X-A_n is $(1,2)^*\alpha_\psi$-open, so by compactness of A_{n1} there is a finite subset $F_0 \subset \Lambda$such that$A_{n1} \subseteq \cap_{n \in F0}(X\text{-}A_n)$. Set $F = \{n_1\} \cup F_0$. Then $\cap_{n \in F} A_n = \phi$.

Lemma 3.11. Let X be a bitopological space, then the following are true.

i) The union of a finite number of $(1,2)^*\alpha_\psi$-compact sets in X is itself $(1,2)^*\alpha_\psi$-compact

ii) If A is a $(1,2)^*\alpha_\psi$-compact subset of X and $B \subseteq A$ is a $(1,2)^*\alpha_\psi$-closed subset of A, then B is $(1,2)^*\alpha_\psi$-compact.

Proof: Part (i) is very trivial, so we prove Part (ii): Consider an $(1,2)^*\alpha_\psi$-open cover $\{U_n : n \in \Lambda\}$ of B. Then $(X\text{-}B) \cup \{U_n : n \in \Lambda\}$ is an $(1,2)^*\alpha_\psi$-open cover of A and hence $A \subseteq (X\text{-}B) \cup (\cap_{n \in F} U_n)$, for some finite collection $F \subset \Lambda$. It follows that $B \subseteq \cup_{n \in F} U_n$, which is proving that B is $(1,2)^*\alpha_\psi$-compact.

Lemma 3.12. Let X be a $(1,2)^*\alpha_\psi$-locally compact Hausdorff space, A is a $(1,2)^*\alpha_\psi$-compact subset of X and G an $(1,2)^*\alpha_\psi$-open subset of X such that $A \subseteq G$. Then there is an $(1,2)^*\alpha_\psi$-open subset V of X with $(1,2)^*\alpha_\psi$-compactclosure $(1,2)^*\alpha_\psi$-cl(V) such that$A \subseteq V \subseteq (1,2)^*\alpha_\psi$-cl(V)$\subseteq G$.

Proof: Since X is $(1,2)^*\alpha_\psi$-locally compact, for each $n \in A$, we can findan $(1,2)^*\alpha_\psi$-open set O_n such that $n \in O_n$ and $(1,2)^*\alpha_\psi$-cl(O_n) is $(1,2)^*\alpha_\psi$-compact. Since $\{O_n : n \in A\}$ is an $(1,2)^*\alpha_\psi$-open cover of A and A is$(1,2)^*\alpha_\psi$-compact, then there is a finite collection $n_1, n_2, \ldots, n_m$ such that $A \subseteq O_{n1} \cup O_{n2} \cup \ldots \cup O_{nm}$. Set $O = O_{n1} \cup O_{n2} \cup \ldots \cup O_{nmm}$ and note that $(1,2)^*\alpha_\psi$-cl(O)$\subseteq (1,2)^*\alpha_\psi$-

cl(O_{n1})* (1,2)*α_ψ-cl(O_{n2})$\cup$ ...$\cup$ (1,2)*α_ψ-cl(O_{nm}). By above Corollary 3.9, (1,2)*α_ψ-cl(O) is (1,2)*α_ψ-compact. This completesthe proof in the case where G=XandV = O. When G $\neq$ X, let C=X G. By Lemma 3.7, for each p$\in$C, there is an (1,2)*α_ψ-open set O'_p such thatA$\subseteq O'_p$ and p$\notin$(1,2)*α_ψ-cl(O'_p). Note that $\cap_{p\in C}$(C$\cap$(1,2)*α_ψ-cl(O)$\cap$ (1,2)*α_ψ-cl(O'_p)) =ϕ, and that each setC$\cap$(1,2)*α_ψ cl(O)) $\cap$(1,2)*α_ψ-cl(O'_p) is (1,2)*α_ψ-compact. By Lemma 3.10, there is a finite setp$_1$, p$_2$, ..., p$_m$, such that $\cap_{i=1}^{m}$ (C$\cap$(1,2)*α_ψ-cl(O)$\cap$(1,2)*α_ψ-cl(O'_{p_i}))=ϕ. SetV=O$\cap$ $O'_{p_1}\cup O'_{p_2}$ $\cup\ldots\cup O'_{p_m}$. Then A$\subseteq$V$\subseteq$ (1,2)*α_ψ-cl(V)$\subseteq$(1,2)*α_ψ-cl(O)$\cap$ (1,2)*α_ψ-cl(O'_{p_1})$\cap$ ...$\cap$ (1,2)*α_ψ-cl(O'_{p_m})$\subseteq$ Gand from Corollary 3.9, (1,2)*α_ψ-cl(V) is(1,2)*α_ψ-compact.

Theorem 3.13. (Baire Category Theorem) If X is a (1,2)*α_ψ-locally compact Hausdorff space, then X is a (1,2)*α_ψ -Baire space

Proof: Let $\{V_i\}_{i=1}^{\infty}$ be a countable collection of (1,2)*α_ψ-open and (1,2)*α_ψ-dense subsets of X. We must show that $\cap_{i=1}^{\infty}$ V$_i$ is (1,2)*α_ψ-dense in X. Consider a non-empty (1,2)*α_ψ-open subset B$_0$ of X. We want to show that ($\cap_{i=1}^{\infty}$V$_i$)$\cap$B$_0$$\neq\phi$. Since V$_1$ is (1,2)*α_ψ-dense in X,B$_0$$\cap$V$_1$$\neq\phi$ and by above Corollary 3.9, (1,2)*α_ψ-cl(B$_0$) is (1,2)*α_ψ-compact. Also we have there is anon-empty (1,2)*α_ψ-open set B$_1$ such that (1,2)*α_ψ-cl(B$_1$) $\subseteq$V$_1$$\cap$B$_0$ and(1,2)*α_ψ-cl(B$_1$) is (1,2)*α_ψ-compact. Since V$_2$ is (1,2)*α_ψ-dense in X we finda non-empty (1,2)*α_ψ-open set B$_2$ such that (1,2)*α_ψ-cl(B$_2$)$\subseteq$V$_2$$\cap$B$_1$$\neq\phi$. Repeat the previous argument for each (1,2)*α_ψ-denseV$_i$, we get a sequenceof non-empty (1,2)*α_ψ-open sets B$_0$, B$_1$, ... such that (1,2)*α_ψ-cl(B$_n$) $\subseteq$ V$_n$$\cap$ B$_{n-1}$$\neq\phi$ $\forall$ n$\geq$1, B$_0$$\supseteq$(1,2)*$\alpha_\psi$-cl(B$_1$)$\supseteq$(1,2)*$\alpha_\psi$-cl(B$_2$)$\supseteq$..., and each (1,2)*$\alpha_\psi$-cl(B$_n$) is (1,2)*$\alpha_\psi$-compact. It follows that $\cap_{n=1}^{\infty}$(1,2)*α_ψ-cl(B$_n$)$\neq\phi$. Since (1,2)*α_ψ-cl(B$_n$) $\subseteq$V$_{n-1}$$\forall$n, we find that $\cap_{n=1}^{\infty}$ (1,2)*α_ψ-cl(B$_n$) $\subseteq$ $\cap_{n=0}^{\infty}$ V$_n$. But $\cap_{n=1}^{\infty}$ (1,2)*α_ψ-cl(B$_n$)$\subseteq$ B$_0$, $\cap_{n=1}^{\infty}$(1,2)*α_ψ-cl(B$_n$)$\subseteq$$\cap_{n=1}^{\infty}V_n$$\cap$B$_0$.

Conclusion

Locally compact Hausdorff space plays important role in the Baire category theorem in General topology. This paper, defined and discussed some more properties of (1,2)*α_ψ-compact sets, (1,2)*α_ψ-locally compact sets and (1,2)*α_ψ-Hausdorff space and their relations. Furthermore, using these concepts we have developed Baire Category Theorem. In future, wemay implement these things into many research areas such as ideal topology, rough set topology, soft topology, fuzzy set topology, digital topology, etc.

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