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On the existence of supremum and infimum of fuzzy random variables

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Abstract

This paper investigates upon the characteristics on the existence of supremum and infimum of fuzzy random variables. This paper provides an improved version of the supremum and infimum of fuzzy random variables which plays a crucial role in the theoretical framework of the same. Finally the existence of supremum and infimum of the level continuous fuzzy valued measurable functions on a closed interval is examined and a necessary and sufficient condition with which its supremum and infimum can be attained, is provided.

Key words: Fuzzy random variables, Fuzzy numbers, Fuzzy valued functions.

1. Introduction

A fuzzy random variable is considered as the imprecise observation of the outcomes in a random experiment. The realization of the coexistence of randomness and fuzziness has ultimately resulted in the formulation of fuzzy random variables. The concept of fuzzy random variable is capable of handling situations where the outcomes of a random experiment are modeled by fuzzy sets. A fuzzy random variable is a mapping which associates a fuzzy set to each element of the universe which is amenable to probability space structure. Fuzzy random variable assigns a fuzzy subset of the final space to each possible outcome of a random experiment. This association exposes the available information about the relation between both the universes. Thus fuzzy random variable is the generalization of the notion of a random variable. But this kind of generalization process has not resulted in a unique fashion. Each formulation of fuzzy random variable has its distinction with the other in the formation of the final space and the manner in which the measurability condition is utilized to this context.

Kwakernaak[6] and Puri and Ralescu[7] have focused on the properties of the multi-valued mappings associated to the α-cuts. Kwakernaak[6] assumes that the outcomes of the fuzzy random variable are fuzzy real subsets and the boundary points of their α-cuts are classical random variables. Puri and Ralescu[7] impose the condition that the α-cuts to be measurable. Klement *et al.*,[4] and Diamond and Kloeden[2] have formulated fuzzy random variables as classical measurable mappings. Kratschmer[5] has revised all these previous formulations and examined some relationships between different measurability conditions.

In this paper Kwakernaak's[6] fuzzy random variables are considered for our investigation. According to Kwakernaak[6] fuzzy random variable is a mapping $X : \Omega \to S$ where Ω is the sample space and S is the space of all piecewise continuous functions $R \to [0,1]$. The mapping X described above characterizes a special type of fuzzy random variable. The random variable U of which this fuzzy random variable is a perception is called an original of the fuzzy random variable. It is pertinent to note that in the case of Kwakernaak's[6] fuzzy random variable, there may exist many originals. For any random variable the acceptability that it is an original is given by the truth value assigned to it, interms of supremum and infinimum of the collection of random variables.

In this paper an investigation is carried upon the characteristic on the existence of supremum and infimum of fuzzy random variables. Finally the existence of supremum and infimum of the level continuous fuzzy valued measurable functions on a closed interval is examined and necessary and sufficient condition with which its supremum and infimum can be attained is provided.

The paper is organized as follows. Section 2 furnishes the necessary technical background. Section 3 deals with the existence of supremum and infimum of fuzzy random variables. In section 4 the notion of level continuity of fuzzy valued measurable functions are introduced. The properties level continuous fuzzy valued measurable functions are derived.

2. Preliminaries :

Let R be the real number field, N be the set of all positive integers and F(R) denote the set of all fuzzy subsets on R which constitutes the fuzzy number space.

For $u \in F(R)$, the α - level set of u is defined as

$$[u]_\alpha = \begin{cases} \{x \in R;\, u(x) \geq \alpha\};\ 0 < \alpha \leq 1 \\ \mathrm{cl}\,\{x \in R;\, u(x) > 0\};\ \text{if}\, \alpha = 0 \end{cases}$$

Definition[3] *: 2.1*

A fuzzy set u on R is called a fuzzy number if it has the following properties

1) u is normal, *i.e* there exists an $x_0 \in R$ such that $u(x_0) = 1$
2) u is convex, *i.e.* $u(\lambda x + (1-\lambda)y) \geq \min \{u(x), u(y)\}$ for $x,y \in R$ and $\lambda \in [0,1]$
3) u is upper semi continuous
4) $[u]_0 = \mathrm{cl}\,\{x \in R \mid u(x) > 0\}$ is a compact set

A real number r can be regarded as the fuzzy number $\tilde{r}$ defined by

$$\tilde{r}(t) = \begin{cases} 1; & t = r \\ 0; & t \neq r \end{cases}$$

It is to be noted that if $u \in F(R)$, then u is a fuzzy number if and only if $[u]_\alpha$ is a non-empty bounded and closed interval for each $\alpha \in [0,1]$. We denote by

$$[u]_\alpha = \left[u^L_\alpha, u^U_\alpha\right], \alpha \in [0,1].$$

A partial ordering in F(R) is defined as $u \le v$ if and only if $u^L_\alpha \le v^L_\alpha$, $u^U_\alpha \le v^U_\alpha$ for all $\alpha \in (0,1]$.

Where $[u]_\alpha = \left[u^L_\alpha, u^U_\alpha\right]$ and $[v]_\alpha = \left[v^L_\alpha, v^U_\alpha\right]$

A subset A of F(R) is said to be bounded above if there exists a fuzzy number M called an upperbound of A such that $u \le M$ for all $u \in A$. A fuzzy number u is called the supremum of A if u is an upper bound of A and $u \le v$ for each upperbound v of A. A lower bound and the infimum of A are defined in a similar fashion. The supremum and the infimum of A are denoted by Sup A and inf A respectively.

Theorem[3] *: 2.1*

Let $u \in F(R)$ and $[u]_\alpha = \left[u^L_\alpha, u^U_\alpha\right]$, then the following conditions are satisfied.

1) U^L_α is a bounded left continuous non decreasing function on (0,1].
2) U^U_α is a bounded left continuous non-increasing function on (0,1]
3) U^L_α and U^U_α are right continuous for each $\alpha = 0$
4) $u^L_1 \le U^U_1$

Conversely if the pair of functions $a(\alpha)$ and $b(\alpha)$ satisfies the conditions (1)–(4) then there exists a unique $u \in F(R)$ such that $[u]_\alpha = [a(\alpha), b(\alpha)]$ for each $\alpha \in [0,1]$

Theorem[3] *: 2.2* For $u, v \in F(R)$ define

$$D(u, v) = \sup_{\alpha \in [0,1]} \max \left\{ \left| u^L_\alpha - v^L_\alpha \right|, \left| u^U_\alpha - v^U_\alpha \right| \right\}$$

then D is a metric on F(R) and (F(R), D) is a complete metric space.

3. Existence Theorem of Supremum and Infimum of Fuzzy Random Variables :

Let (Ω, A, P) be a probability space. $R(\Omega)$ denotes the set of all random variables on (Ω, A, P)

Let $RI(\Omega) = \{ \tilde{x} ; \tilde{x} = [x^L, x^U], x^L, x^U \in R(\Omega), x^L \le x^U$ every where on $\Omega\}$

The elements in $RI(\Omega)$ are called closed random interval numbers on (Ω, A, P). If $x \in R(\Omega)$ then $x = [x, x] \in RI(\Omega)$. Hence $R \subset R(\Omega) \subset RI(\Omega)$ where $R = (-\infty, \infty)$. The following theorem deals with the existence of supremum and infimum of fuzzy random variables.

Theorem : 3.1 :

Let (Ω, A, P) be a probability space. Let A be a non empty subset of $RI(\Omega)$. If A has an upper bound, then its supremum $u \in R(\Omega)$ must exist and has the following expressions.

$$u = \sup A = \bigcup_{\alpha\in(0,1]} \alpha\left[\sup_{X\in A} X_\alpha^L, \sup_{X\in A} X_\alpha^U\right] \quad (3.1)$$

$$\bigcup_\alpha^L = \sup_{X\in A} X_\alpha^L, \bigcup_\alpha^U = \inf_{r<\alpha} \sup_{X\in A} X_r^U \text{ for each } \alpha \in (0,1] \quad (3.2)$$

$$\bigcup_o^L = \inf_{\alpha>0} \sup_{X\in A} X_\alpha^L, \ \bigcup_0^U = \sup_{X\in A} X_0^U \quad (3.3)$$

Dually if A has a lower bound then its infimum $V \in R(\Omega)$ must exist and has the following expressions :

$$V = \inf A = \bigcup_{\alpha\in(0,1]} \alpha\left[\inf_{X\in A} X_\alpha^L, \inf_{X\in A} X_\alpha^U\right] \quad (3.4)$$

$$V_\alpha^L = \sup_{r<\alpha} \inf_{X\in A} X_r^L, V_\alpha^U = \inf_{X\in A} X_\alpha^U \text{ for each } \alpha \in (0,1] \quad (3.5)$$

$$V_0^L = \inf_{X\in A} X_0^L, \ V_0^U = \sup_{\alpha>0} \inf_{X\in A} X_\alpha^U \quad (3.6)$$

Proof :

Suppose that $M\in RI(\Omega)$ be an upper-bound of A.

Then $X \leq M$ for all $X \in A$. So we have

$$X_\alpha^L \leq M_\alpha^L, \ X_\alpha^U \leq M_\alpha^U \text{ for each } \alpha\in[0,1]$$

It is easy to see that for each fixed $\alpha \in [0,1]$, the sets of real numbers $\{X_\alpha^L; X \in A\}$ and $\{X_\alpha^U; X \in A\}$ are bounded above. Hence we can define the interval

$$H(\alpha) = \left[\sup_{X\in A} X_\alpha^L, \sup_{X\in A} X_\alpha^U\right]$$

Obviously $0 < \alpha < \mu \leq 1$ implies $H(\mu) \subset H(\alpha)$.

By the representation theorem of fuzzy sets, there exists a fuzzy set U on R such that

$$U = \bigcup_{\alpha\in(0,1]} \alpha H(\alpha) \text{ and}$$

$$[u]_\alpha = \bigcap_{r<\alpha} H(r)$$

$$= \bigcap_{r<\alpha}\left[\sup_{X\in A} X_r^L, \sup_{X\in A} X_r^U\right]$$

$$= \left[\sup_{r<\alpha} \sup_{X\in A} X_r^L, \ \inf_{r<\alpha} \sup_{X\in A} X_r^U\right] \quad (3.7)$$

for each $\alpha \in (0, 1]$.

Moreover $[u]_0$ is a closed interval. In fact for each $\alpha \in (0,1]$ and $r\in(0, \alpha)$ we have

$$[u]_\alpha \subset \left[\sup_{X\in A} X_r^L, \sup_{X\in A} X_r^U\right]$$

$$\subset \left[\inf_{\mu>0} \sup_{X\in A} X_\mu^L, \sup_{\mu>0} \sup_{X\in A} X_\mu^U\right]$$

This implies that

$$[u]_0 = \mathrm{cl}\left(\bigcup_{\alpha\in(0,1]} [u]_\alpha\right)$$

$$\subset \left[\inf_{\mu>0} \sup_{X\in A} X_\mu^L, \ \sup_{\mu>0} \sup_{X\in A} X_\mu^U\right]$$

$\bigcup_{\alpha\in(0,1]} [U]_\alpha$ is nothing but an interval.

Hence $[u]_0$ is a bounded closed interval.

$\therefore u \in RI(\Omega)$.

From expression (3.7) and by theorem 2.1 that

$$u_\alpha^L = \sup_{r<\alpha} \sup_{X\in A} X_r^L = \sup_{X\in A} X_r^L$$

$$U_\alpha^U = \inf_{r<\alpha} \sup_{X\in A} X_r^U \quad \text{for } \alpha \in (0,1]$$

$$U_0^L = \lim_{\alpha\to 0^+} u_\alpha^L = \inf_{\alpha>0} \sup_{X\in A} X_\alpha^L$$

$$U_0^U = \lim_{\alpha\to 0^+} u_\alpha^U = \sup_{\alpha>0} \inf_{r<\alpha} \sup_{X\in A} X_\alpha^U$$

(3.2) and the first formula of (3.3) are established.

To prove $u_0^U = \sup_{X\in A} X_0^U$, it is enough if we prove that

$$\sup_{\alpha>0} \inf_{r<\alpha} \sup_{X\in A} X_r^U = \sup_{X\in A} X_0^U$$

obviously $\sup_{X\in A} X_0^U \geq \sup_{\alpha>0} \inf_{r<\alpha} \sup_{X\in A} X_\alpha^U$

Since X_α^U is non-increasing and right-continuous at $\lambda = 0$

We have $\sup_{X\in A} X_0^U = \sup_{X\in A} \sup_{\alpha>0} X_\alpha^U = \sup_{\alpha>0} \sup_{X\in A} X_\alpha^U$

$$\leq \sup_{\alpha>0} \inf_{r<\alpha} \sup_{X\in A} X_r^U$$

This establishes the second formula of (3.3)

Lastly we prove that u is the supremum of A

i.e u = sup A

From (3.2) it can be deduced that

$$X_\alpha^L \leq U_\alpha^L,\ X_\alpha^U \leq U_\alpha^U \quad \text{for each } \alpha \in (0,1]$$

and $X \in A$

In fact $X_\alpha^L \leq \sup_{X\in A} X_\alpha^L = u_\alpha^L$

Moreover for any $r \in (0, \alpha)$ we have

$$X_\alpha^U \leq X_r^U \leq \sup_{X\in A} X_\alpha^U \text{ and so}$$

$$X_\alpha^U \leq \inf_{r<\alpha} \sup_{X\in A} X_\alpha^L = U_\alpha^U$$

This shows that u is an upper bound of A. On the other hand suppose that $W \in RI(\Omega)$ is also an upperbound of A.

Then we have $\sup_{X\in A} X_\alpha^L \leq W_\alpha^L$ and

$\sup_{X\in A} X_\alpha^U \leq W_\alpha^U$ for all $\alpha \in (0,1]$ and so

$$u_\alpha^L \leq W_\alpha^L$$

$$U_\alpha^U = \inf_{r<\alpha} \sup_{X\in A} X_r^U \leq \inf_{r<\alpha} W_r^U = W_\alpha^U;$$

$\alpha \in (0,1]$

Hence $U \leq W$. This shows that u is the least upper bound of A.

$\therefore$ u = sup A

Suppose that $L \in RI(\Omega)$ be a lower bound of A. Then $X \geq L$ for all $X \in A$. So we have

$$X_\alpha^L \geq L_\alpha^L,\ X_\alpha^U \geq L_\alpha^U \quad \text{for each } \alpha \in [0,1].$$

It is easy to see that for each fixed $\alpha \in [0,1]$ the sets of real numbers $\{X_\alpha^L, X \in A\}$ and $\{X_\alpha^U, X \in A\}$ are bounded below. Hence we can define the interval

$$J(\alpha) = \left[\inf_{X\in A} X_\alpha^L, \inf_{X\in A} X_\alpha^U\right]$$

Obviously $0 < \alpha < \mu \leq 1$ implies $J(\mu) \subset J(\alpha)$

By the representation theorem on fuzzy sets, there exists a fuzzy set V on R such that

$V = \bigcap_{\alpha\in(0,1]} \alpha\, J(\alpha)$ and

$$[V]_\alpha = \bigcup_{r<\alpha} J(r)$$

$$= \bigcup_{r<\alpha} \left[\inf_{X\in A} X_r^L , \inf_{X\in A} X_r^U\right]$$

$$= \left[\inf_{r<\alpha} \inf_{X\in A} X_r^L , \inf_{r<\alpha} \inf_{X\in A} X_r^U\right]$$

for each $\alpha \in (0,1]$

Moreover $[v]_0$ is a closed interval. Infact for each $\alpha \in (0,1]$ and $r\in(0, \alpha)$ we have

$$[v]_\alpha \subset \left[\inf_{X\in A} X_r^L , \inf_{X\in A} X_r^U\right]$$

$$\subset \left[\inf_{\mu>0} \inf_{X\in A} X_\mu^L , \sup_{\mu>0} \inf_{X\in A} X_r^U\right] \quad (3.8)$$

This shows that

$$[v]_0 = \mathrm{cl}\left(\bigcap_{\alpha\in(0,1]} [v]_\alpha\right)$$

$$\subset \left[\inf_{\mu>0} \inf_{X\in A} X_\mu^L , \sup_{\mu>0} \inf_{X\in A} X_r^U\right]$$

$\bigcap_{\alpha\in(0,1]} [v]_\alpha$ is nothing but an interval

Hence $[v]_0$ is a bounded closed interval

$\therefore\ V \in RI(\Omega)$

From expression (3.8) and by theorem 2.1 that

$$V_\alpha^L = \sup_{r<\alpha} \inf_{X\in A} X_r^L$$

$$V_\alpha^U = \inf_{r<\alpha} \inf_{X\in A} X_r^U = \inf_{X\in A} X_\alpha^U \text{ for each } \alpha \in (0,1]$$

$$V_0^L = \lim_{\alpha\to 0+} U_\alpha^U = \sup_{\alpha>0} \inf_{r<\alpha} \inf_{X\in A} X_r^L$$

$$V_0^U = \lim_{\alpha\to 0+} U_\alpha^L = \sup_{\alpha>0} \inf_{X\in A} X_r^U$$

(3.5) and the second formula of (3.6) are established.

To prove $V_0^L = \inf_{X\in A} X_0^L$ it is enough if we prove that

$$\sup_{\alpha>0} \inf_{r<\alpha} \inf_{X\in A} X_r^L = \inf_{X\in A} X_0^L$$

obviously $\inf_{X\in A} X_0^L = \sup_{\alpha>0} \inf_{r<\alpha} \inf_{X\in A} X_r^U$

Since X_α^L is non-decreasing and left continuous at $\lambda = 0$

We have $\inf_{X\in A} X_0^L = \inf_{X\in A} \sup_{\alpha>0} X_\alpha^L$

$$= \sup_{\alpha>0} \inf_{X\in A} X_\alpha^L$$

$$\geq \sup_{\alpha>0} \inf_{r<\alpha} \inf_{X\in A} X_r^U$$

This establishes the second formula of (3.5)

Lastly we prove that V is the infimum of A

i.e V = inf A

From (3.5) it can be deduced that

$$X_\alpha^L \geq V_\alpha^L$$

$X_\alpha^U \geq V_\alpha^U$ for each $\alpha \in (0,1]$ and $X \in A$

Infact $X_\alpha^L \geq \inf_{X \in A} X_\alpha^L = V_\alpha^L$

Moreover for any $r \in (0, \alpha)$ we have

$$X_\alpha^L \geq X_r^L \geq \inf_{X \in A} X_r^U \text{ and so}$$

$$X_\alpha^L \leq \inf_{r<\alpha} \inf_{X \in A} X_r^L = V_\alpha^L$$

This shows that V is an upperbound of A. On the otherhand suppose that $V_1 \in RI(\Omega)$ is also a lower bound of A.

Then we have $\inf_{X \in A} X_\alpha^L \geq (V_1)_\alpha^L$ and

$\inf_{X \in A} X_\alpha^U \geq (V_1)_\alpha^U$ for all $\alpha \in (0,1]$ and so

$$V_\alpha^L \geq (V_1)_\alpha^L$$

$$V_\alpha^U = \sup_{r<\alpha} \inf_{X \in A} X_r^U$$

$$\geq \sup_{r<\alpha} (V_1)_\alpha^U = (V_1)_\alpha^U \ ; \ \alpha \in (0,1]$$

Hence $V \geq V_1$ this shows that V is the greatest lower bound of A.

$\therefore$ V = sup A.

4. Some Properties of Level Continuous Fuzzy Valued Measurable Functions:

In this section the concept of fuzzy valued measurable function and the level continuity of fuzzy valued measurable function are introduced. Some interesting properties of level continuous fuzzy valued measurable functions are established. Let X be a non-empty set.

Definition : 4.1

Let F(R) be the set of all fuzzy number. $(F(R))_{cl}$ denotes the set of all closed fuzzy numbers, $(F(R))_b$ denotes the set of all bounded fuzzy numbers and $(F(R))_s$ denotes the set of all standard fuzzy numbers. We say that

(i) $\tilde{f}(x)$ is a fuzzy valued function if $\tilde{f} : X \to F(R)$

(ii) $\tilde{f}(x)$ is a closed fuzzy valued function if $\tilde{f} : X \to (F(R))_{cl}$

(iii) $\tilde{f}(x)$ is a bounded fuzzy valued function if $\tilde{f} : X \to (F(R))_b$

(iv) $\tilde{f}(x)$ is a standard fuzzy valued function if $\tilde{f} : X \to (F(R))_s$

We denote $\tilde{f}_\alpha^L(x) = \left(\tilde{f}(x)\right)_\alpha^L$ and

$$\tilde{f}_\alpha^U(x) = \left(\tilde{f}(x)\right)_\alpha^U$$

Definition 4.2 :

By a fuzzy valued measure $\tilde{\mu}$ on a measure space (X, M) we mean a non-negative fuzzy valued set function defined for all sets of M and satisfying the following two conditions.

(i) $\tilde{\mu}(\emptyset) = \tilde{0}$

(ii) $\tilde{\mu}\left(\bigcup_{n=1}^{\infty} E_n\right) = \bigoplus_{n=1}^{\infty} \tilde{\mu}(E_n)$ exists for any sequence E_i of disjoint measurable sets.

$\tilde{\mu}$ is called closed (bounded or standard) fuzzy valued measure of $\tilde{\mu}$ is a non-negative closed (bounded or standard) fuzzy valued set function.

Definition 4.3 :

Let (X, M) be a measurable space and (R, B) be a measurable space.

(i) $f : X \to P(R)$ (power set of R) is a set valued function then f is called measurable if and only if $\{(x, y); y \in f(x)\}$ is a measurable subset of $M \times B$ [1].

(ii) $\tilde{f}$ is a fuzzy valued function then $\tilde{f}_\alpha$ is a set valued function for each α. $\tilde{f}$ is called measurable if and only if $\tilde{f}_\alpha$ is measurable for each α.[8]

(iii) Let $\tilde{f}$ be a closed fuzzy valued function. $\tilde{f}$ is called strongly measurable if and only if $\tilde{f}$ is measurable and one of $\tilde{f}_\alpha^L$ and $\tilde{f}_\alpha^U$ is measurable for each α [8].

A fuzzy valued function $\tilde{f} : X \to F(R)$ is said to be continuous at $t_0 \in X$ if for each $\varepsilon > 0$ there is a $\delta > 0$ such that $D(\tilde{f}(t), \tilde{f}(t_0)) < \varepsilon$ whenever $t \in X$ with $|t - t_0| < \delta$. If $\tilde{f}(t)$ is continuous at each $t \in X$ then we say that $\tilde{f}(t)$ is continuous on X.

Definition 4.4 :

A fuzzy valued function $\tilde{f} : X \to F(R)$ is said to be level-continuous at $t_0 \in X$ if

$$\lim_{t \to t_0} \tilde{f}(t) = \tilde{f}(t_0) \text{ i.e.,}$$

$$\lim_{t \to t_0} \left(\tilde{f}(t)\right)_\alpha^L = \left(\tilde{f}(t_0)\right)_\alpha^L$$

$$\lim_{t \to t_0} \left(\tilde{f}(t)\right)_\alpha^U = \left(\tilde{f}(t_0)\right)_\alpha^U \text{ for each } \alpha \in (0,1]$$

If $\tilde{f}$ is level continuous at each $t \in X$ then we say that $\tilde{f}$ is level continuous on X.

Theorem 4.1:

Let $\tilde{f} : X \to F(R)$ be a fuzzy valued measurable function, and level continuous at $t_0 \in X$ and $\alpha \in (0,1]$ be given.

Let $\tilde{f} : X \to F(R)$ be a fuzzy valued measurable function, then for any $\{t_n\} \subset X$ with $t_n \to t_0$ and any $\alpha_n \subset (0, \alpha)$ with $\alpha_n \to \alpha$ we have lim $n \to \infty$ we have lim $n \to \infty$ $\left(\tilde{f}(t_n)\right)_{\alpha_n}^U = \left(\tilde{f}(t_0)\right)_\alpha^U$

Proof : By stipulation $\tilde{f}$ is level continuous at t_0 and $\{t_n\} \subset X$ with $t_n \to t_0$. we have lim $n \to \infty$ $\tilde{f}(t_n) = \tilde{f}(t_0) \in F(R)$.

By theorem 4.1 we knowthat

$\left(\tilde{f}(t_n)\right)_\alpha^U$ is eventually equi-left continuous at $\alpha \in (0,1]$ and so for any $\varepsilon > 0$ there exists a natural number N_1 and $\delta > 0$ ($\delta < \alpha$) such that

$$\left|\left(\tilde{f}(t_n)\right)_\mu^U - \left(\tilde{f}(t_n)\right)_\alpha^U\right| < \varepsilon/2 \text{ for all } \mu \in (\alpha-\delta, \alpha] \text{ and } n \geq N_1 \quad (4.1)$$

since $\alpha_n \in (0, \alpha)$ with $\alpha_n \to \alpha$ and $\left(\tilde{f}(t_n)\right)_\alpha^U \to \left(\tilde{f}(t_0)\right)_\alpha^U$ there exists a natural number N_2 such that

$$\alpha_n \in (\alpha-\delta, \alpha) \text{ and } \left|\left(\tilde{f}(t_n)\right)_\alpha^U - \left(\tilde{f}(t_0)\right)_\alpha^U\right| < \varepsilon/2 \text{ for all } n \geq N_2$$

Let $N = \max\{N_1, N_2\}$. By (4.1) and (4.2) we have

$$\left|\left(\tilde{f}(t_n)\right)_{\alpha_n}^U - \left(\tilde{f}(t_0)\right)_\alpha^U\right| \leq \left|\left(\tilde{f}(t_n)\right)_{\alpha_n}^U - \left(\tilde{f}(t_n)\right)_\alpha^U\right| + \left|\left(\tilde{f}(t_n)\right)_{\alpha_n}^U - \left(\tilde{f}(t_0)\right)_\alpha^U\right| < \varepsilon \text{ for all } n \geq N$$

$$\therefore \lim n \to \infty \left[\tilde{f}(t_n)\right]_{\alpha_n}^U = \left(\tilde{f}(t_0)\right)_\alpha^U$$

Theorem 4.2:

Let $\tilde{f} : X \to F(R)$ be level-continuous fuzzy valued measurable function on X. Then we have

(1) $\sup t \in X \left[\tilde{f}(t)\right]_\alpha^L = \sup \mu < \alpha \left(\tilde{f}(t)\right)_\mu^L$ for each $\alpha \in (0,1]$

(2) $\sup t \in X \left[\tilde{f}(t)_\alpha^U\right] = \inf_{\mu<\alpha} \sup_{t\in X} \left(\tilde{f}(t)\right)_\mu^U$ for each $\lambda \in (0,1]$

(3) $\sup t \in X \left(\tilde{f}(t)\right)_0^L = \inf r > 0 \sup_{t\in X} \left[\tilde{f}(t)\right]_r^L$

(4) $\sup t \in X \left(\tilde{f}(t)\right)_0^U = \sup r > 0 \sup_{t\in X} \left[\tilde{f}(t)\right]_r^U$

Proof :

(1) Since $\left(\tilde{f}(t)\right)_\alpha^L$ is non-decreasing and left continuous for α

we have $\sup_{t\in X} \left(\tilde{f}(t)\right)_\alpha^L = \sup_{t\in X} \sup_{\mu<\alpha} \left(\tilde{f}(t)\right)_\mu^L$

$$= \sup_{\mu<\alpha} \sup_{t\in X} \left(\tilde{f}(t)\right)_\mu^L$$

(2) Let $\sup_{t\in X} \left[\tilde{f}(t)\right]_\alpha^U = p$

$\inf_{\mu<\alpha} \sup_{t\in X} \left(\tilde{f}(t)\right)_\mu^U = q$

Since $\left(\tilde{f}(t)\right)_\mu^U$ is non-increasing for α

We have $\left(\tilde{f}(t)\right)_\mu^U \leq \left(\tilde{f}(t)\right)_\mu^U \leq \sup_{t\in X} \left(\tilde{f}(t)\right)_\mu^U$

for all $\mu < \alpha$ and $t \in [a,b]$. This shows that $p \leq q$.

Now suppose that $p < q$. Taking a fixed $c \in (p,q)$ and $\mu_n \in (0,\alpha)$ with $\mu_n \to \alpha$ then we have

$$C < q \le \sup_{t\in X} \left(\tilde{f}(t)\right)^U_{\mu_n}$$ and so there exists $t_n \in X$ such that $\left(\tilde{f}(t_n)\right)^U_{\mu} > C$. without loss of generality we can assume $t_n \to t_0 \in X$. By theorem 4.1.

We infer that

$$p < C \le \sup_{t\in X} \left(\tilde{f}(t_0)\right)^U_{\alpha} \le \sup_{t\in X} \left(\tilde{f}(t)\right)^U_{\alpha} = p$$

which is a contradiction.

$\therefore$ p = q.

$$(3) \sup_{t\in X} \left(\tilde{f}(t)\right)^U_0 = \lim_{\alpha\to 0^+} \sup_{t\in X} \left(\tilde{f}(t)\right)^L_{\alpha}$$

$$= \inf_{r<\alpha} \sup_{t\in X} \left(\tilde{f}(t)\right)^L_{r}$$

$$= \inf_{r>0} \sup_{t\in X} \left(\tilde{f}(t)\right)^L_{r}$$

$$(4) = \sup_{t\in X} \left(\tilde{f}(t)\right)^U_0 = \lim_{\alpha\to 0^+} \sup_{t\in X} \left(\tilde{f}(t)\right)^U_{\alpha}$$

$$= \sup_{\alpha>0} \inf_{r<\alpha} \sup_{t\in X} \left(\tilde{f}(t)\right)^U_{\alpha}$$

$$= \sup_{r>0} \sup_{t\in X} \left(\tilde{f}(t)\right)^U_{r}$$

Theorem 4.3 :

Let $\tilde{f} : A \to F(R)$ be a level continuous fuzzy valued measurable function.

Then u = $\sup_{t\in A} \tilde{f}(t)$ must exist in F(R) and for each $\alpha \in [0,1]$

$$u^L_{\alpha} = \sup_{t\in A} \left(\tilde{f}(t)\right)^L_{\alpha}$$

$$U^U_{\alpha} = \sup_{t\in A} \left(\tilde{f}(t)\right)^U_{\alpha}$$

Proof :

Since $\tilde{f}$ is level continuous on A, it is easy to see that for each fixed $\alpha \in [0,1]$, $\left(\tilde{f}(t)\right)^L_{\alpha}$ and $\left(\tilde{f}(t)\right)^U_{\alpha}$ are continuous in A. Thus we can define two functions a(α), b (α) on [0,1] by

$$a(\alpha) = \sup_{t\in A} \left(\tilde{f}(t)\right)^L_{\alpha}$$

$$b(\alpha) = \sup_{t\in A} \left(\tilde{f}(t)\right)^U_{\alpha}$$

obviously a(α) is non-decreasing, b(α) is non-increasing and $\alpha(1) \le \beta(1)$. It follows from (1) and (2) of theorem 4.2 that

$$a(\alpha) = \sup_{\mu<\alpha} \sup_{t\in A} \left(\tilde{f}(t)\right)^L_{\mu}$$

$$= \sup_{\mu<\alpha} a(\mu) = \lim_{\mu\to\alpha^-} a(\mu)$$

and $$b(\alpha) = \inf_{\mu<\alpha} \sup_{t\in A} \left(\tilde{f}(t)\right)^U_{\mu}$$

$$= \inf_{\mu<\alpha} b(\mu) = \inf_{\mu<\alpha} b(\mu) \text{ for each } \alpha \in (0,1]$$

This implies that a(α) and b(α) are left continuous at each $\alpha \in (0,1]$ similarly by using (3) and (4) of theorem 4.2. We can prove that a(α) and b(α) are right continuous at α=0.

This shows that a(α) and b(α) satisfy the condition (1) – (4) of theorem 2.1. Hence there exists a $W \in F(R)$ such that

$$W_\alpha^L = a(\alpha)$$

$$W_\alpha^U = b(\alpha) \text{ for each } \alpha \in [0,1]$$

It is easy to see that $\tilde{f}(t) \le W$ for all $t \in A$. i.e. W is an upperbound for the set of all fuzzy valued measurable functions. Thus by theorem 3.1 we infer that the supremum exists in F(R) and write

$$u = \sup_{t\in A} \tilde{f}(t)$$

By (3.2) and (3.3) and theorem 4.2 it is easy to prove that (1) holds. This completes the proof.

Theorem 4.4 :

Let $\tilde{f} : A \to F(R)$ be level continuous fuzzy valued measurable function. Then can attain its supremum in A if and only if the $\sup_{t\in A} \tilde{f}(t)$ has the level –approximation property, that is there exists $\{t_n\} \in A$ such that lim $n\to\infty$ $\tilde{f}(t_n) = \sup_{t\in A} \tilde{f}(t)$.

Proof :

We assume that $\tilde{f}$ attains its supremum in A.

Suppose that there exists $t_0 \in A$ such that

$$\tilde{f}(t_0) = \sup_{t\in A} \tilde{f}(t)$$

Taking $t_n = t_0$, n = 1, 2, …… we have

$$\lim_{n\to\infty} \tilde{f}(t_n) = \sup_{t\in A} \tilde{f}(t)$$

conversely we assume that

$$\lim_{n\to\infty} \tilde{f}(t_n) = \sup_{t\in A} \tilde{f}(t). \text{ and}$$

prove that $\tilde{f}$ attains its supremum in A.

Suppose that $\sup_{t\in A} \tilde{f}(t)$ has the level approximation property.

i.e. there exists $\{t_n\} \subset A$ such that

$$\lim_{n\to\infty} \tilde{f}(t_n) = \sup_{t\in[a,b]} \tilde{f}(t)$$

Without loss of generality we can assume $t_n \to t_0 \in A$

Since $\tilde{f}$ is level continuous at t_0 we have

$$\left(\tilde{f}(t_n)\right)_\alpha^L \to \left(\tilde{f}(t_0)\right)_\alpha^L \text{ and}$$

$$\left(\tilde{f}(t_n)\right)_\alpha^U \to \left(\tilde{f}(t_0)\right)_\alpha^U \text{ for each } \alpha\in[0,1]$$

Thus we have

$$\left| \left[\sup_{t\in A} \tilde{f}(t)\right]_\alpha^L - \left[\tilde{f}(t_0)\right]_\alpha^L \right|$$

$$\le \left| \left[\sup_{t\in A} \tilde{f}(t)\right]_\alpha^L - \left[\tilde{f}(t_n)\right]_\alpha^L \right| + \left| \left[(\tilde{f}(t_n)\right]_\alpha^L - \left[\tilde{f}(t_0)\right]_\alpha^L \right|$$

$$\to 0 \text{ as } n \to \infty.$$

and so $\left[\sup_{t\in A} \tilde{f}(t)\right]_{\alpha}^{L} = \left[\tilde{f}(t_0)\right]_{\alpha}^{L}$. Similarly we have

$$\left[\sup_{t\in A} \tilde{f}(t)\right]_{\alpha}^{U} = \left[\tilde{f}(t_0)\right]_{\alpha}^{U} \text{ for each}$$

$\alpha \in [0,1]$

Hence $\sup_{t\in A} \tilde{f}(t) = \tilde{f}(t_0)$.

References :

1. R. J. Aumann, Integrals of set valued functions, *J. Math. Anal. Appl. 12,* 1-12, (1965).
2. P. Diamond, P. Kloeden, Metric Spaces of fuzzy sets, World Scientific Singapore, (1994).
3. R. Goetschel, W. Voxman, Elementary fuzzy calculus, *Fuzzy sets and systems, 18,* 31-42, (1986).
4. E.P. Klement, M.L. Puri, D.A. Ralescu, Limit theorems for fuzzy random variables, *Proc. Roy. Soc. London, A 407,* 171-182, (1986).
5. V. Kratschmer, A unified approach to fuzzy random variables, *Fuzzy sets and systems 123,* 1-9, (2001).
6. H. Kwakernaak, Fuzzy random variables : Definition and theorems, *Inform. Sci., 15,* 1-29, (1978).
7. M. Puri, D. Ralescu, Fuzzy random variables, *J. Math. Anal. Appl.114,* 409-422 (1986).
8. H.C. Wu, Fuzzy valued integrals of fuzzy valued measurable functions with respect to fuzzy valued measures based on closed intervals, *Fuzzy sets and systems 87,* 65-78, (1997).