Ultra Scientist Vol. 24(1)A, 59-62 (2012).

Order-dimensional topological spaces

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(Acceptance Date 24th November, 2011)

Abstract

In is paper order – dimensional topological spaces are introduced. Several basic properties relating to order – dimension defined in terms of order of elements are established. The definitions of order of an element and of a set in a topological space are taken from[4] and [5] respectively.

Key words: order of an element, order of a set, order of a topology, finite order dimensional, order topological dimension

Subject Classification : AMS (2000) : 54 A 99.

Introduction

The order of collection of subsets of a topological space has been defined[3]. Finite dimensional topological spaces with their topological dimensions are also discussed[3]. The idea of an order-dimensional topological space stems from these concepts mentioned[3].

Definition 1: Let (X,τ) be a topological space and $a \in X$. Then *order of a relative to topology* τ, $O_\tau(a)$ is the number of τ-open sets containing a.

Definition 2: Let (X,τ) be a topological space and $A \subset X$. Then *order of A relative to topology* τ, $O_\tau(A)$ is the number of τ-open sets containing A.

Definition 3: A topology τ on a set X is said to have *order* if there exist an element $a \in X$ such that $O_\tau(a) = m + 1$ and for no $b \in X$, $O_\tau(b) > m + 1$.
Equivalently, a topology τ on a set X is said to have *order m* denoted by $O(\tau)$ if highest order

attained by any element of the space is m+1.

Remark: $O(\tau)$ will be infinite if order of some element of the space is infinite

Definition 4: A topological space (X,τ) is said to be *finite order – dimensional* if there is some non – negative integer such that $O_\tau(a) \leq m+1$ for every $a \in X$.

The *order topological dimension* of X denoted by $dim_\tau^o X$ is defined to be the smallest value of m for which above statement holds.

The following results are direct consequences of definitions.

Theorem 1 : If (X,τ) is a topological space such that $O(\tau)=m$, then for any topology τ' coarser then τ, $O(\tau') \leq m$.

Theorem 2: Let (X,τ) be a topological space and $A \subset X$ such that A intersects every open set of X in a different set. Then order of τ is same as order of τ_A provided there exists an element in A whose τ -order is $O(\tau)+1$.

Proof: By theorem[4] 5,

$O_\tau(a) = O_{\tau_A}(a)$ for all $a \in A$.
Let $O(\tau) = m$.
Now there exists $a \in A$ such that
$O_\tau(a) = m+1$
or $O_{\tau_A}(a) = m+1$
And $O_\tau(b) = O_{\tau_A}(b) \leq m+1$ for all $b \in A$.
Hence $O(\tau_A) = m$.

Example 1: Let $X=\{a,b,c,d,e\}$,
$A=\{a,d,e\}$,
$\tau = \{X, \emptyset, \{a\}, \{c,d\}, \{a,c,d\}, \{b,c,d,e\}\}$
Then $\tau_A=\{A, \emptyset, \{a\}, \{d\}, \{a,d\}, \{d,e\}\}$
Clearly $O(\tau) = O(\tau_A) = 3$.

Theorem 3: If (X,τ) and (Y,υ) are two topological spaces then

$$O(\tau \times \upsilon) \leq O(\tau).O(\upsilon) + O(\tau) + O(\upsilon)$$

where $\tau \times \upsilon$ is product topology on $X \times Y$.

Theorem 4: If (X,τ) is a topological space such that $dim_\tau^o X$ is finite then $dim_\tau^o A$ and $dim_{\tau_A}^o A$ are finite for $A \subset X$.

Theorem 5: (X,τ) is a topological space and if $dim_\tau^o X$ is infinite then both $O(\tau)$ and X are infinite.

Theorem 6: (X,τ) is a topological space and A is subspace of X then $dim_{\tau_A}^o X \leq dim_\tau^o A$.

Theorem 7: (X,τ) is a topological space and Y intersect τ-open sets of X in distinct sets then $dim_\tau^o Y$ is infinite if $dim_\tau^o X$ is infinite.

Proof: By theorem 2,

$$dim_\tau^o X = dim_{\tau_Y}^o Y = \infty .$$

Moreover by theorem 6,

$$dim_{\tau_Y}^o Y \leq dim_\tau^o Y$$

and hence $dim_\tau^o Y$ is infinite.

Example 2: Let X be an infinite set

and $Y \subset X$ such that both are Y and $X \sim Y$ infinite.

Let $\tau = \{\emptyset, \text{all supersets of } Y\}$ be a topology on X.

Then $dim_\tau^o X = \infty$.

Also τ_Y is indiscrete and hence $dim_{\tau_Y}^o Y$ is zero but $dim_\tau^o Y$ is infinite.

Theorem 8: Subspace of a finite order – dimensional topological space is finite order-dimensional.

Corollary 1: Intersection of two finite order – dimensional topological spaces is finite order – dimensional.

Theorem 9: Product of two finite order-dimensional topological spaces is finite order-dimensional.

Proof: Follows from theorem 3 and definition 4.

Theorem 10: Let Y and Z be disjoint finite order – dimensional subspaces of a topological space (X,τ) such that (i) $X = Y \cup Z$, (ii) No finite number of open sets of X intersects Y or Z in same set. Then X is finite order – dimensional and $dim_\tau^o X = max\ \{dim_{\tau_Y}^o Y, dim_{\tau_Z}^o Z\}$.

Proof: Let $dim_{\tau_Y}^o Y = n$ and $dim_{\tau_Z}^o Z = m$. Without loss of generality, let $m \le n$.

By theorem[4] 5

$O_{\tau_Y}(x) = O_\tau(x)$ for $x \in Y$

$O_{\tau_Z}(x) = O_\tau(x)$ for $x \in Z$

Now there exists $x \in X$ such that $O_\tau(x) = O_{\tau Y}(x) = n+1$.

and for any $a(\neq x) \in X$, $O_\tau(a) \le n+1$. Hence X is finite order–dimensional and

$dim_\tau^o X = n$

$= max\ \{dim_{\tau_Y}^o Y , dim_{\tau_Z}^o Z\}$.

There are abundant number of spaces of finite order-dimensional for e.g. finite subspaces of an infinite order-dimensional spaces are all finite order-dimensional.

Theorem 11: Homeomorphic image of finite order-dimensional space is finite order –dimensional.

Proof: Let $f : (X,\tau) \to (Y,\upsilon)$ be a homeomorphism and let be finite order-dimensional. Then there exists non – negative integer *m* such that

$O_\tau(x) \le m+1$ for every $x \in X$

Let $y \in Y$ then there is $x \in X$ such that $f(x)=y$. By theorem[4] 7,

$O_\tau(x) = O_\upsilon(f(x)) = O_\upsilon(y)$.

Hence $O_\upsilon(y) \le m+1$ for every $y \in Y$ or Y is finite order – dimensional.

Example 3: Order – dimensional of any indiscrete space is zero.

Example 4: Order – dimension of any discrete space with n elements is $2^{n-1} - 1$.

Example 5: Let $X = \mathbb{N}$ and

$\tau = \{ A_m : m \in \mathbb{N} \}$

where $A_m = \{m, m+1, \ldots\ldots\}$

Let $Y = \{1, 2, \dots, n\}$

Then $\tau_Y = \emptyset, Y, \{2,3, \dots, n\}, \dots, \{n\}$ and $O_{\tau_Y}(a)$ is either n or $n-1, \cdots$ or 1 for any $a \in Y$.

Hence $dim^o_{\tau_Y} Y = n - 1$ although $dim^o_\tau X = \infty$

Example 6: Let (X,τ) be a topological space where τ is $\{\emptyset, \text{all supersets of a subset } A \text{ of } X\}$.

Then τ_A is indiscrete and $dim^o_{\tau_A} A = 0$

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