Ultra Scientist Vol. 24(1)A, 67-80 (2012).

On Weakly ϕ-Ricci Symmetric Lorentzian Para (LP) α-Sasakian Manifolds

S. S. PUJAR[1] And V. J. KHAIRNAR[2]

[1]Department of Mathematics, K. K. Wagh Institute of Engineering Education and Research, Panchavati, Nasik-422 003, M. S. (INDIA)
E-mail.: drsspujar@rediffmail.com
[2]N. V. P. Mandal's Arts, Commerce and Science College, Lasalgaon, Tal. Niphad, Dist. Nasik-422 306, M. S. (INDIA)
E-mail.: vjkhairnar@rediffmail.com

(Acceptance Date 24th November, 2011)

Abstract

The purpose of the paper deals with weakly ϕ-Ricci Symmetric; locally ϕ- Ricci Symmetric and ϕ- Ricci recurrent Lorentzian Para (LP) α-Sasakian manifolds of dimension $\left(M^{2n+1}, g\right)$ (n>1). The existence of weakly ϕ-Ricci Symmetric Lorentzian Para (LP) α-Sasakian manifolds is ensured by an example.

1. Introduction

The notion of Lorentzian manifold was first introduced by K. Motsumoto[2] in 1989. The same was independently studied by I.Mihai and Rosca[7] and obtained the several results of Lorentzian Para Sasakian manifold. Some of the other authors have also studied LP- Sasakian manifolds such as U.C. De[9], Motsumoto and I. Mihai[3], A. A. Shaikh[8], Abolfazl Taleshian and Nader Asghar[1], Lovjoy Das,[4,5], Mobin Ahmad and Janardhan Ojha[6]. S. S. Pujar[10] have introduced the notion of Lorentzian Para α -Sasakian Manifolds which is generalised from of the LP-Sasakian manifold and series of basic results.

In this paper, we introduce the notion of weakly ϕ- Ricci Symmetric Lorentzian Para α-Sasakian manifold as follows.

Definition1.1 A Lorentzian Para α-Sasakian manifold $\left(M^{2n+1}, g\right)$ (n>1) is said to be weakly ϕ- Ricci Symmetric if its Ricci tensor Q of type (1, 1) is not identically zero and satisfies the condition:

$$\phi^2\left(\nabla_X Q\right)(Y) = A(X)Q(Y) + B(Y)Q(X) + g(QX, Y)\rho , \tag{1.1}$$

where g(X, ρ)=C(X), Q is the Ricci operator such that

$$g(QX, Y) = S(X, Y),$$

for all vector fields X, Y in M^{2n+1}, A, B and C are non zero 1-forms, called the associated 1-forms of the manifold and ∇ denotes the operator of the covariant differentiation with respect to the metric tensor g, ϕ is a tensor field of type (1, 1) and S is the Ricci curvature tensor of type (0, 2).

The objective of the paper is to study weakly ϕ Ricci Symmetric; locally ϕ Ricci Symmetric and ϕ- Ricci recurrent Lorentzian Para (LP) α-Sasakian manifolds. Section 2 deals with preliminaries of Lorentzian Para (LP) α-Sasakian manifolds. In section 3 we have found that expressions for A(X) + B(X) – C(X). As a geometrical meaning of the 1-form A, it is proved that *"If a weakly Lorentzian Para (LP) α-Sasakian manifold* $\left(M^{2n+1}, g\right)$ *(n>1) is ϕ Ricci Symmetric with α is constant, then the relation*

$$A(\phi X) = 0$$

holds."

In section 4 of the paper, it is proved that if a weakly Lorentzian Para α-Sasakian manifold is locally Ricci symmetric, then we have found that expressions for A(X)+B(X)-C(X). In section 5, it is proved that if a weakly Lorentzian Para α-Sasakian manifold is locally ϕ Ricci recurrent, then we have found expressions for B(X)-C(X). As a geometrical meaning of the 1-forms B and C, it is proved that *"If a weakly Lorentzian Para (LP) α-Sasakian manifold* $\left(M^{2n+1}, g\right)$ *(n>1) is ϕ Ricci Symmetric with α is constant, then the relation*

$$B(\phi X) - C(\phi X) = 0$$

holds."

In section 6, we have provided an concrete example for the existence of weakly ϕ Ricci Symmetric Lorentzian Para (LP) α-Sasakian manifolds.

2.*Lorentzian Para (LP) α-Sasakian manifolds:*

For a almost Lorentzian contact manifold M of dimension 2n+1, we have

$$\cdot\, \phi^2 X = X + \eta(X)\xi, \eta(X) = g(X, \xi) \text{ ,(2.1)}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y) \text{. (2.2)}$$

For a c^{∞} vector field X on M and ϕ is a tensor field of type (1, 1), ξ is a characteristic vector field and η is 1- form. From these conditions, one can deduce that

$$\phi(\xi) = 0, \eta(\xi) = -1 .$$

An almost contact metric structure[1,4,5] is called a Lorentzian Para Sasakian manifold if

$$(\nabla_X \phi)(Y) = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi \text{ ,}$$

where ∇ is the Livi- civita connection with respect to g. Using above formula, one can deduce

More generally, S. S. Pujar[10] introduces the notion of Lorentzian Para -Sasakian Manifold as follows.

Definition 2.1: A Manifold M with Lorentzian almost contact metric structure (ϕ, ξ, η, g) is said to be the Lorentzian Para

α-Sasakian Manifold if

$$(\nabla_X\phi)Y = \alpha\{g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi\}, \tag{2.3}$$

where α is a smooth function on M.
From (2.3) it follows that

$$\nabla_X\xi = \alpha\phi(X) \tag{2.4}$$

$$(\nabla_X\eta)(Y) = \alpha g(\phi X, Y) \tag{2.5}$$

$$R(X,Y)\xi = \alpha^2\{\eta(Y)X - \eta(X)Y\} + \{(X\alpha)\phi(Y) - (Y\alpha)\phi(X)\} \tag{2.6}$$

$$R(\xi,Y)\xi = \alpha^2\{Y + \eta(Y)\xi\} + (\xi\alpha)\phi(Y), \quad R(\xi,\xi)\xi = 0 \tag{2.7}$$

$$R(\xi,Y)X = \alpha^2\{g(X,Y)\xi - \eta(X)Y\} - (X\alpha)\phi(Y) + g(\phi X, Y)(\operatorname{grad}\alpha) \tag{2.8}$$

$$S(Y,\xi) = 2n\alpha^2\eta(Y) - \{(Y\alpha)\omega + \phi(Y)\alpha\} \tag{2.9}$$

$$S(\xi,\xi) = -2n\alpha^2 - (\xi\alpha)\omega \tag{2.10}$$

$$Q\xi = 2n\alpha^2\xi - (\operatorname{grad}\alpha)\omega - \phi(\operatorname{grad}\alpha), \tag{2.11}$$

where R is the curvature tensor of type (1, 3) of the manifold and S is the Ricci curvature on M. For any vector field Y on M and $\omega = g(\phi e_i, e_i)$, Q is the symmetric endomorphism of the tangent space at each point of the manifold corresponding to the Ricci tensor S and g(QX, Y)=S(X, Y) for all vector fields X and Y on M.

Definition 2.2. A weakly Lorentzian Para (LP) α-Sasakian manifolds $\left(M^{2n+1}, g\right)$ (n>1) is called Locally ϕ-Ricci Symmetric if its Ricci tensor Q of type (1,1) is not identically zero and satisfies the condition:

$$\phi^2\left(\nabla_X Q\right)(Y) = 0, \tag{2.12}$$

where X, Y are vector fields on M.

If B=C=0, then weakly Lorentzian Para (LP) α-Sasakian reduces to ϕ-Ricci recurrent. Hence we define:

Definition 2.3. A weakly Lorentzian Para (LP) α-Sasakian manifolds $\left(M^{2n+1}, g\right)$ (n>1) is called ϕ-Ricci Recurrent if its Ricci tensor Q is not identically zero and satisfies the condition:

$$\phi^2\left(\nabla_X Q\right)(Y) = A(X)Q(Y), \tag{2.13}$$

where A is associated 1-form and X, Y are any vector fields on M.

3. *Weakly ϕ- Ricci Symmetric Lorentzian Para (LP) α-Sasakian manifolds:*

In this section some theorems on weakly ϕ- Ricci Symmetric Lorentzian Para (LP) α-Sasakian manifolds are studied.

Theorem 3.1 If a weakly Lorentzian Para (LP) α-Sasakian manifold $\left(M^{2n+1}, g\right)$ (n>1) is ϕ-Ricci Symmetric, then the relation

$$A(\xi) + B(\xi) + C(\xi) = \frac{-4n\alpha(\xi\alpha) - \eta(\xi(\operatorname{grad}\alpha)\omega) - \eta((\operatorname{grad}\alpha)(\xi\omega))}{-2n\alpha^2 - (\xi\alpha)\omega} \tag{3.1}$$

holds provided that

$$2n\alpha^2 + (\xi\alpha)\omega \neq 0 \tag{3.2}$$

Proof: Using (2.1) in (1.1), we get

$$\left(\nabla_X Q\right)(Y) + \eta\left(\left(\nabla_X Q\right)(Y)\right)\xi = A(X)Q(Y) + B(Y)Q(X) + g(QY, X)\rho \tag{3.3}$$

Taking inner product on both sides with respect to the vector field V, we find

$$g\left(\left(\nabla_X Q\right)(Y), V\right)+\eta\left(\left(\nabla_X Q\right)(Y)\right)g(\xi, V)$$
$$= A(X)g(Q(Y), V)+B(Y)g(Q(X), V)$$
$$+g(QY, X)g(\rho, V) \qquad (3.4)$$

Taking V to be orthogonal to ξ in (3.4) and using the fact that

g(QX, V)=S(X, V) and g(ρ, V)=C(V), we have

$$g\left(\left(\nabla_X Q\right)(Y), V\right)=A(X)S(Y, V)$$
$$+B(Y)S(X, V)+C(V)S(Y, X) \qquad (3.5)$$

Substituting X=Y=V=ξ in (3.5), we get

$$g\left(\left(\nabla_\xi Q\right)(\xi), \xi\right)=A(\xi)S(\xi, \xi)+B(\xi)S(\xi, \xi)+C(\xi)S(\xi, \xi)$$
$$=\left\{A(\xi)+B(\xi)+C(\xi)\right\}S(\xi, \xi)$$
$$=\left\{A(\xi)+B(\xi)+C(\xi)\right\}\left(-2n\alpha^2-(\xi\alpha)\omega\right)$$
$$(3.6)$$

The left hand side of (3.6) is given by

$$g\left(\nabla_\xi Q(\xi)-Q\nabla_\xi\xi, \xi\right)$$
$$=g\left(\nabla_\xi\left(Q\xi, \xi\right)\right)$$
$$=\nabla_\xi\left(2n\alpha^2\right)g(\xi, \xi)-g\left(\nabla_\xi\phi\left(\mathrm{grad}\alpha\right), \xi\right)$$
$$-g\left(\nabla_\xi\left(\mathrm{grad}\alpha\right)\omega, \xi\right)$$
$$=-4n\alpha\left(\xi\alpha\right)-\eta\left(\xi\left(\mathrm{grad}\alpha\right)\omega\right)$$
$$-\eta\left(\left(\mathrm{grad}\alpha\right)\left(\xi\omega\right)\right) \qquad (3.7)$$

Substituting this on left hand side of (3.6), we get

$$\left\{A(\xi)+B(\xi)+C(\xi)\right\}\left(-2n\alpha^2-\left(\xi\alpha\right)\omega\right)$$
$$=-4n\alpha(\xi\alpha)-\eta\left(\xi\left(\mathrm{grad}\alpha\right)\omega\right)$$
$$-\eta\left(\left(\mathrm{grad}\alpha\right)\left(\xi\omega\right)\right) \qquad (3.8)$$

Further simplifying (3.8), we get (3.1) of Theorem 3.1, provided that (3.2), the proof of the Theorem 3.1 completes.

Corollary 3.1 If a weakly Lorentzian Para (LP) α-Sasakian manifold $\left(M^{2n+1}, g\right)$ (n>1) is ϕ-Ricci Symmetric and satisfies (3.2) and α is constant, then the relation

$$A(\xi)+B(\xi)+C(\xi)=0$$

holds.

Proof: Follows from Theorem 3.1.

Theorem 3.2 If a weakly Lorentzian Para (LP) α-Sasakian manifold $\left(M^{2n+1}, g\right)$ (n>1) is ϕ- Ricci Symmetric and satisfies (3.2), then the relation

$$A(X)=\frac{4n\alpha(X\alpha)+\eta\left(X\left(\mathrm{grad}\alpha\right)\omega\right)+\eta\left(\left(\mathrm{grad}\alpha\right)(X\omega)\right)+\eta\left(X\left(\phi\mathrm{grad}\alpha\right)\right)}{2n\alpha^2+(\xi\alpha)\omega}$$

$$+\frac{\alpha\left[\left(\phi X\alpha\right)\omega+\left(\phi^2X\right)\alpha\right]}{2n\alpha^2+(\xi\alpha)\omega}$$

$$\left[\frac{4n\alpha(\xi\alpha)+\eta\left(\xi\left(\mathrm{grad}\alpha\right)\omega\right)+\eta\left(\left(\mathrm{grad}\alpha\right)\left(\xi\omega\right)\right)}{\left(2n\alpha^2+\left(\xi\alpha\right)\omega\right)^2}\right.$$

$$\left.-\frac{A(\xi)}{2n\alpha^2+\left(\xi\alpha\right)\omega}\right]\left[2n\alpha^2\eta(X)-(X\alpha)\omega-\phi(X)\alpha\right]$$

$$(3.9)$$

holds.

Proof: Substituting Y=V=ξ in (3.5), we get

$$g\left(\left(\nabla_X Q\right)(\xi),\xi\right) = A(X)S(\xi,\xi) + B(\xi)S(X,\xi) + C(\xi)S(\xi,X) \quad (3.10)$$

Now consider by the linearity property of Ricci tensor S

$$S\left(\alpha\phi(X),\xi\right) = \alpha S\left(\phi X,\xi\right)$$

$$S\left(\alpha\phi(X),\xi\right) = \alpha\left[2n\alpha^2\eta(\phi X) - \xi\left(\phi X\alpha\right)\omega - \left(\phi^2 X\right)\alpha\right] \quad (3.11)$$

Hence left hand side of (3.10) reduces to

$$g\left(\nabla_X Q(\xi) - Q\nabla_X\xi,\xi\right) = -4n\alpha\left(X\alpha\right) - \eta\left(X\left(\mathrm{grad}\alpha\right)\omega\right) - \eta\left(\left(\mathrm{grad}\alpha\right)\left(X\omega\right)\right) - \eta\left(X\left(\phi\mathrm{grad}\alpha\right)\right) - \alpha\left[2n\alpha^2\eta(\phi X) - \xi\left(\phi X\alpha\right)\omega - \left(\phi^2 X\right)\alpha\right] \quad (3.12)$$

Also the right hand side of (3.10) is

$$A(X)S(\xi,\xi) + B(\xi)S(X,\xi) + C(\xi)S(\xi,X) = A(X)\left[-2n\alpha^2 - (\xi\alpha)\omega\right] + \left[B(\xi)+C(\xi)\right]\left[2n\alpha^2\eta(X) - X\alpha)\omega - \phi(X)\alpha\right]$$

Also from (3.1), we get

$$A(X)S(\xi,\xi) + B(\xi)S(X,\xi) + C(\xi)S(\xi,X) = A(X)\left[-2n\alpha^2 - \left(\xi\alpha\right)\omega\right] + \left[\frac{4n\alpha(\xi\alpha) + \eta\left(\xi\left(\mathrm{grad}\alpha\right)\omega\right) + \eta\left(\left(\mathrm{grad}\alpha\right)\left(\xi\omega\right)\right)}{2n\alpha^2 + (\xi\alpha)\omega}\right] - A(\xi)\left]\left[2n\alpha^2\eta(X) - (X\alpha)\omega - \phi(X)\alpha\right] \quad (3.13)$$

from (3.12) and (3.13) it follows that

$$4n\alpha\left(X\alpha\right) + \eta\left(X\left(\mathrm{grad}\alpha\right)\omega\right) + \eta\left(\left(\mathrm{grad}\alpha\right)X\omega\right) + \eta\left(X\left(\phi\mathrm{grad}\alpha\right)\right) + \alpha\left[2n\alpha^2\eta\left(\phi X\right) - \xi\left(\phi X\alpha\right)\omega - \left(\phi^2 X\right)\alpha\right]$$
$$= A(X)\left[2n\alpha^2 + \left(\xi\alpha\right)\omega\right] - \left[\frac{4n\alpha(\xi\alpha) + \eta\left(\xi\left(\mathrm{grad}\alpha\right)\omega\right) + \eta\left(\left(\mathrm{grad}\alpha\right)\xi\omega\right)}{2n\alpha^2 + (\xi\alpha)\omega}\right] - A(\xi)\left]\left[2n\alpha^2\eta(X) - (X\alpha)\omega - \phi(X)\alpha\right] \quad (3.14)$$

Simplifying (3.14) for A(X), this completes the proof of the Theorem 3.2.

Theorem 3.3: If a weakly Lorentzian Para (LP) α-Sasakian manifold $\left(M^{2n+1},g\right)$ (n>1) is ϕ-Ricci Symmetric and satisfies (3.2), then the relation

$$B(Y) = \frac{-g\left(\nabla_\xi Q(Y),\xi\right) + \left[2n\alpha^2\eta\left(\nabla_\xi Y\right) - \left(\nabla_\xi Y\alpha\right)\omega + \phi\left(\nabla_\xi Y\right)\alpha\right]}{2n\alpha^2 + \left(\xi\alpha\right)\omega} + \left[\frac{4n\alpha(\xi\alpha) + \eta\left(\xi\left(\mathrm{grad}\alpha\right)\omega\right) + \eta\left(\left(\mathrm{grad}\alpha\right)\xi\omega\right)}{(2n\alpha^2 + \left(\xi\alpha\right)\omega)^2} - \frac{B(\xi)}{2n\alpha^2 + \left(\xi\alpha\right)\omega}\right]\left[2n\alpha^2\eta(Y) - (Y\alpha)\omega - \phi(Y)\alpha\right] \quad (3.15)$$

holds.

Proof: Substituting X=V=ξ in (3.5), we get

$$g\left(\nabla_\xi Q(Y),\xi\right) = A(\xi)S(Y,\xi) + B(Y)S(\xi,\xi) + C(\xi)S(Y,\xi) \quad (3.16)$$

Left hand side of (3.16) is

$$g\left(\nabla_\xi Q(Y) - Q\nabla_\xi Y,\xi\right) = g\left(\nabla_\xi\left(QY\right),\xi\right) - S\left(\nabla_\xi Y,\xi\right) = g\left(\nabla_\xi\left(QY\right),\xi\right) - \left[2n\alpha^2\eta\left(\nabla_\xi Y\right)\right.$$

$$-\left(\nabla_\xi Y\alpha\right)\omega-\phi\left(\nabla_\xi Y\right)\alpha\Big]$$

Right hand side of (3.16) is

$$\{A(\xi)+C(\xi)\}S(Y,\xi)+B(Y)S(\xi,\xi)$$

$$=\left[\frac{4n\alpha(\xi\alpha)+\eta\left(\xi\left(\text{grad}\alpha\right)\omega\right)+\eta\left(\left(\text{grad}\alpha\right)\xi\omega\right)}{2n\alpha^2+\left(\xi\alpha\right)\omega}\right.$$

$$\left.-B(\xi)\right]\left[2n\alpha^2\eta(Y)-\left(Y\alpha\right)\omega-\phi(Y)\alpha\right]$$

$$+B(Y)\left[-2n\alpha^2-\left(\xi\alpha\right)\omega\right]$$

from (3.16) equating LHS and RHS, we get

$$g\left(\nabla_\xi(QY),\xi\right)-\left[2n\alpha^2\eta\left(\nabla_\xi Y\right)-\left(\nabla_\xi Y\alpha\right)\omega-\phi\left(\nabla_\xi Y\right)\alpha\right]$$

$$=\left[\frac{4n\alpha(\xi\alpha)+\eta\left(\xi\left(\text{grad}\alpha\right)\omega\right)+\eta\left(\left(\text{grad}\alpha\right)\xi\omega\right)}{2n\alpha^2+(\xi\alpha)\omega}\right.$$

$$\left.-B(\xi)\right]\left[2n\alpha^2\eta(Y)-(Y\alpha)\omega-\phi(Y)\alpha\right]$$

$$+B(Y)\left[-2n\alpha^2-\left(\xi\alpha\right)\omega\right] \qquad (3.17)$$

Simplifying (3.17), we get (3.15). This completes the proof of the Theorem 3.3.

Theorem 3.4 : If a weakly Lorentzian Para (LP) α-Sasakian manifold $\left(M^{2n+1},g\right)$ $(n>1)$ is ϕ-Ricci Symmetric and satisfies (3.2), then the relation

$$C(V)=\frac{-g\left(\nabla_\xi(QV),\xi\right)+\left[2n\alpha^2\eta\left(\nabla_\xi V\right)-\left(\nabla_\xi V\alpha\right)\omega-\phi\left(\nabla_\xi V\right)\alpha\right]}{2n\alpha^2+\left(\xi\alpha\right)\omega}$$

$$+\left[\frac{4n\alpha(\xi\alpha)+\eta\left(\xi\left(\text{grad}\alpha\right)\omega\right)+\eta\left(\left(\text{grad}\alpha\right)\xi\omega\right)}{(2n\alpha^2+\left(\xi\alpha\right)\omega)^2}\right.$$

$$\left.-\frac{C(\xi)}{2n\alpha^2+\left(\xi\alpha\right)\omega}\right]\left[2n\alpha^2\eta(V)-(V\xi)\omega-\phi(V)\alpha\right] \qquad (3.18)$$

holds.

Proof: Similar arguments as in the above Theorem 3.3 the proof of the Theorem 3.4 follows.

Theorem 3.5 If a weakly Lorentzian Para (LP) α-Sasakian manifold $\left(M^{2n+1},g\right)$ $(n>1)$ is ϕ-Ricci Symmetric and satisfies (3.2), then the relation

$$B(X)-C(X)=\left[C(\xi)-B(\xi)\right]\left[\frac{2n\alpha^2\eta(X)-(X\alpha)\omega-\phi(X)\alpha}{2n\alpha^2+\left(\xi\alpha\right)\omega}\right] \qquad (3.19)$$

holds.

Proof: Replacing Y and V by X in (3.15) and (3.18), then taking the difference and simplifying, we get

$$B(X)-C(X)=\frac{-g\left(\nabla_\xi(QX),\xi\right)+\left[2n\alpha^2\eta\left(\nabla_\xi X\right)-\left(\nabla_\xi X\alpha\right)\omega-\phi\left(\nabla_\xi X\right)\alpha\right]}{2n\alpha^2+\left(\xi\alpha\right)\omega}$$

$$+\left[\frac{4n\alpha\left(\xi\alpha\right)+\eta\left(\xi\left(\text{grad}\alpha\right)\omega\right)+\eta\left(\left(\text{grad}\alpha\right)\xi\omega\right)}{(2n\alpha^2+\left(\xi\alpha\right)\omega)^2}\right.$$

$$-\frac{B(\xi)}{2n\alpha^2+(\xi\alpha)\omega}\Bigg]\left[2n\alpha^2\eta(X)-(X\alpha)\omega-\phi(X)\alpha\right]$$

$$+\frac{g\left(\nabla_\xi(QX),\xi\right)-\left[2n\alpha^2\eta\left(\nabla_\xi X\right)-\left(\nabla_\xi X\alpha\right)\omega-\phi\left(\nabla_\xi X\right)\alpha\right]}{2n\alpha^2+(\xi\alpha)\omega}$$

$$-\Bigg[\frac{4n\alpha(\xi\alpha)+\eta\left(\xi\left(\mathrm{grad}\alpha\right)\omega\right)+\eta\left(\left(\mathrm{grad}\alpha\right)\xi\omega\right)}{(2n\alpha^2+(\xi\alpha)\omega)^2}$$

$$-\frac{C(\xi)}{2n\alpha^2+(\xi\alpha)\omega}\Bigg]\left[2n\alpha^2\eta(X)-(X\alpha)\omega-\phi(X)\alpha\right] \tag{3.20}$$

Now simplifying further (3.20), the proof of Theorem 3.5 follows.

Theorem 3.6 If a weakly Lorentzian Para (LP) α-Sasakian manifold $\left(M^{2n+1},g\right)$ (n>1) is ϕ-Ricci Symmetric and satisfies (3.2), then the relation

$$A(X)+B(X)-C(X)=\frac{4n\alpha(X\alpha)+\eta\left(X\left(\mathrm{grad}\alpha\right)\omega\right)+\eta\left(\left(\mathrm{grad}\alpha\right)X\omega\right)+\eta\left(X\left(\phi\mathrm{grad}\alpha\right)\right)}{2n\alpha^2+(\xi\alpha)\omega}$$

$$+\frac{\alpha\left[\xi\left(\phi X\alpha\right)\omega+\left(\phi^2X\right)\alpha\right]}{2n\alpha^2+(\xi\alpha)\omega}$$

$$+\left[\frac{4n\alpha(\xi\alpha)+\eta\left(\xi\left(\mathrm{grad}\alpha\right)\omega\right)+\eta\left(\left(\mathrm{grad}\alpha\right)\xi\omega\right)}{\left(2n\alpha^2+(\xi\alpha)\omega\right)^2}\right]\left[2n\alpha^2\eta(X)-(X\alpha)\omega-\phi(X)\alpha\right]$$

$$+\left[\frac{C(\xi)-B(\xi)-A(\xi)}{2n\alpha^2+(\xi\alpha)\omega}\right]\left[2n\alpha^2\eta(X)-(X\alpha)\omega-\phi(X)\alpha\right] \tag{3.21}$$

holds.

Proof: Follows by taking the addition of (3.9) and (3.19). This completes the proof of the Theorem 3.6.

If α is constant, then the relation (3.19) of Theorem 3.5 is becomes that

$$B(X) - C(X) = \left[C(\xi) - B(\xi)\right]\eta(X) \quad (3.22)$$

If α is constant, then the relation (3.21) of Theorem 3.6 is becomes that

$$A(X) + B(X) - C(X) = \left[C(\xi) - B(\xi) - A(\xi)\right]\eta(X) \quad (3.23)$$

Using (3.22) in (3.23) we get

$$A(X) = -A(\xi)\eta(X) \quad (3.24)$$

And putting X=ϕX in (3.24), we get

$$A(\phi X) = 0 .$$

Remark: As a geometrical meaning of the 1-form A, it is proved that "If a weakly Lorentzian Para (LP) α-Sasakian manifold $\left(M^{2n+1}, g\right)$ (n>1) is ϕ Ricci Symmetric with α is constant, then the relation

$$A(\phi X) = 0$$

holds.

4. Locally ϕ- Ricci Symmetric Spaces :

Lemma 4.1 If a weakly Lorentzian Para (LP) α-Sasakian manifold $\left(M^{2n+1}, g\right)$ (n>1) is locally ϕ-Ricci symmetric and satisfies (3.2), then the relation

$$A(\xi) + B(\xi) + C(\xi) = 0 \quad (4.1)$$

Proof. Suppose manifold M is weakly ϕ Ricci Symmetric Lorentzian Para (LP) α-Sasakian. Then from equation (1.1), we get

$$A(X)Q(Y) + B(Y)Q(X) + g(QY, X)\rho = 0 \quad (4.2)$$

Taking inner product with respect to the vector field V, we find

$$A(X)g(Q(Y), V) + B(Y)g(Q(X), V) + g(QY, X)g(\rho, V) = 0 \quad (4.3)$$

Using the fact that

g (QX, V)=S(X, V) and g(ρ, V)=C(V)

in (4.3), we have

$$A(X)S(Y, V) + B(Y)S(X, V) + C(V)S(Y, X) = 0 \quad (4.4)$$

Substituting X=Y=V=ξ in (4.4), we get

$$A(\xi)S(\xi,\xi) + B(\xi)S(\xi,\xi) + C(\xi)S(\xi,\xi) = 0 \quad (4.5)$$

Using (2.10) for S (ξ,ξ) in (4.5), we get (4.1).

Theorem 4.2 If a weakly Lorentzian Para (LP) α-Sasakian manifold $\left(M^{2n+1}, g\right)$ (n>1) is locally ϕ-Ricci Symmetric and satisfies (3.2), then the relation

$$A(X) = \left(\frac{-A(\xi)}{2n\alpha^2 + (\xi\alpha)\omega}\right)\left[2n\alpha^2\eta(X) - (X\alpha)\omega - \phi(X)\alpha\right] \quad (4.6)$$

holds.

Proof : Substituting Y=V=ξ in (4.4), we get

$$A(X)S(\xi,\xi) + B(\xi)S(X,\xi) + C(\xi)S(\xi, X) = 0 \quad (4.7)$$

Using (2.10) for S(ξ,ξ) and (2.9) for S(ξ, X) in (4.7), we get

$$A(X)\left[-2n\alpha^2 - (\xi\alpha)\omega\right] + \left[B(\xi) + C(\xi)\right]\left[2n\alpha^2\eta(X) - X\alpha)\omega - \phi(X)\alpha\right] = 0 \quad (4.8)$$

Using (4.1) for B (ξ) +C (ξ) in (4.8) and after simplification, we get (4.6). This completes the proof of the Theorem 4.2.

Theorem 4.3 If a weakly Lorentzian Para (LP) α-Sasakian manifold $\left(M^{2n+1}, g\right)$ (n>1) is locally ϕ-Ricci Symmetric and satisfies (3.2), then the relation

$$B(Y) = \left(\frac{-B(\xi)}{2n\alpha^2 + (\xi\alpha)\omega}\right)\left[2n\alpha^2\eta(Y) - (Y\alpha)\omega - \phi(Y)\alpha\right] \tag{4.9}$$

holds.

Proof: Substituting X=V=ξ in (4.4), we get

$$\{A(\xi) + C(\xi)\}S(Y,\xi) + B(Y)S(\xi,\xi) = 0 \tag{4.10}$$

Using (2.10) for S (ξ,ξ), (2.9) for S (Y,ξ) in (4.10), we get

$$B(Y)\left[-2n\alpha^2 - (\xi\alpha)\omega\right] + \left(A(\xi)\right.$$

$$\left. + C(\xi)\right)\left[2n\alpha^2\eta(Y) - (Y\alpha)\omega - \phi(X)\alpha\right] = 0 \tag{4.11}$$

Using (4.1) for A(ξ)+C(ξ) in (4.11) and after simplification, we get (4.9). This completes the proof of the Theorem 4.3.

Theorem 4.4 If a weakly Lorentzian Para (LP) α-Sasakian manifold $\left(M^{2n+1}, g\right)$ (n>1) is locally ϕ-Ricci Symmetric and satisfies (3.2), then the relation

$$C(V) = \left(\frac{-C(\xi)}{2n\alpha^2 + (\xi\alpha)\omega}\right)\left[2n\alpha^2\eta(V) - (V\xi)\omega - \phi(V)\alpha\right] \tag{4.12}$$

holds.

Proof: Similar arguments as in the above Theorem 4.3 the proof of the Theorem 4.4 follows.

Theorem 4.5 If a weakly Lorentzian Para (LP) α-Sasakian manifold $\left(M^{2n+1}, g\right)$ (n>1) is locally ϕ-Ricci Symmetric and satisfies (3.2), then the relation

$$B(X) - C(X) = \left[C(\xi) - B(\xi)\right]\left[\frac{2n\alpha^2\eta(X) - (X\alpha)\omega - \phi(X)\alpha}{2n\alpha^2 + (\xi\alpha)\omega}\right] \tag{4.13}$$

holds.

Proof. Replacing Y and V by X in (4.9) and (4.12), and then taking the difference, we get

$$B(X) - C(X) = \left(\frac{-B(\xi)}{2n\alpha^2 + (\xi\alpha)\omega}\right)\left[2n\alpha^2\eta(X) - (X\alpha)\omega - \phi(X)\alpha\right]$$

$$-\left(\frac{-C(\xi)}{2n\alpha^2 + (\xi\alpha)\omega}\right)\left[2n\alpha^2\eta(X) - (X\xi)\omega - \phi(X)\alpha\right] \tag{4.14}$$

after simplification (4.14), we get (4.13). This completes the proof of the Theorem 4.5.

Theorem 4.6 If a weakly Lorentzian Para (LP) α-Sasakian manifold $\left(M^{2n+1}, g\right)$ (n>1) is locally ϕ-Ricci Symmetric and satisfies (3.2), then the relation

$$A(X)+B(X)-C(X) = \left[\frac{C(\xi) - B(\xi) - A(\xi)}{2n\alpha^2 + (\xi\alpha)\omega}\right]$$

$$\left[2n\alpha^2\eta(X) - (X\alpha)\omega - \phi(X)\alpha\right] \tag{4.15}$$

holds.

Proof: Follows by taking the addition

of (4.6) and (4.13). This completes the proof of the Theorem 4.6.

If α is constant, then the relation (4.13) of Theorem 4.5 becomes that

$$B(X)-C(X)=\left[C(\xi)-B(\xi)\right]\eta(X) \quad (4.16)$$

If α is constant, then the relation (4.15) of Theorem 4.6 is becomes that

$$A(X)+B(X)-C(X)=\left[C(\xi)-B(\xi)-A(\xi)\right]\eta(X) \quad (4.17)$$

Using (4.16) in (4.17) we get

$$A(X)=-A(\xi)\eta(X) \quad (4.18)$$

And putting $X=\phi X$ in (4.18), we get

$$A(\phi X)=0.$$

Remark: As a geometrical meaning of the 1-form A, it is proved that "If a weakly Lorentzian Para (LP) α-Sasakian manifold $\left(M^{2n+1},g\right)$ (n>1) is ϕ Ricci Symmetric with α is constant, then the relation

$$A(\phi X)=0$$

holds.

5. Recurrent Spaces :

Lemma 5.1 If a weakly Lorentzian Para (LP) α-Sasakian manifold $\left(M^{2n+1},g\right)$ (n>1) is ϕ-Ricci recurrent and satisfies (3.2), then the relation

$$B(\xi)+C(\xi)=0 \quad (5.1)$$

Proof. Suppose manifold M is weakly Lorentzian Para (LP) α-Sasakian with ϕ-Ricci recurrent. Then from equation (1.1), we get

$$B(Y)Q(X)+g(QY,X)\rho=0 \quad (5.2)$$

Taking inner product with respect to the vector field V, we find

$$B(Y)g(Q(X),V)+g(QY,X)g(\rho,V)=0 \quad (5.3)$$

Using the fact that

$g(QX, V)=S(X, V)$ and $g(\rho, V)=C(V)$ in (5.3), we have

$$B(Y)S(X,V)+C(V)S(Y,X)=0 \quad (5.4)$$

Substituting $X=Y=V=\xi$ in (5.4), we get

$$B(\xi)S(\xi,\xi)+C(\xi)S(\xi,\xi)=0 \quad (5.5)$$

Using (2.10) for $S(\xi,\xi)$ in (5.5), we get (5.1).

Theorem 5.2 If a weakly Lorentzian Para (LP) α-Sasakian manifold $\left(M^{2n+1},g\right)$ (n>1) is ϕ-Ricci recurrent[12] and satisfies (3.2), then the relation

$$B(Y)=\left(\frac{-B(\xi)}{2n\alpha^2+(\xi\alpha)\omega}\right)\left[2n\alpha^2\eta(Y)-(Y\alpha)\omega-\phi(Y)\alpha\right] \quad (5.6)$$

holds.

Proof: Substituting $X=V=\xi$ in (5.4), we get

$$C(\xi)S(Y,\xi)+B(Y)S(\xi,\xi)=0 \quad (5.7)$$

Using (2.10) for $S(\xi,\xi)$, (2.9) for $S(Y,\xi)$ in (5.7), we get

$$B(Y)\left[-2n\alpha^2-(\xi\alpha)\omega\right]+\left(C\left(\xi\right)\right)\left[2n\alpha^2\eta(Y)-(Y\alpha)\omega-\phi(X)\alpha\right]=0 \quad (5.8)$$

Using (5.1) for $C(\xi)$ in (5.8) and after simplification, we get (5.6). This completes the proof of the theorem 5.2.

Theorem 5.3 If a weakly Lorentzian Para (LP) α-Sasakian manifold $\left(M^{2n+1}, g\right)$ (n>1) is ϕ-Ricci recurrent and satisfies (3.2), then the relation

$$C(V)=\left(\frac{-C(\xi)}{2n\alpha^2+(\xi\alpha)\omega}\right)\left[2n\alpha^2\eta(V)-(V\xi)\omega-\phi(V)\alpha\right] \quad (5.9)$$

holds.

Proof: Similar arguments as in the above Theorem 5.2 the proof of the Theorem 5.3 follows.

Theorem 5.4 If a weakly Lorentzian Para (LP) α-Sasakian manifold $\left(M^{2n+1}, g\right)$ (n>1) is ϕ-Ricci recurrent and satisfies (3.2), then the relation

$$B(X)-C(X)=\left[C(\xi)-B(\xi)\right]\left[\frac{2n\alpha^2\eta(X)-(X\alpha)\omega-\phi(X)\alpha}{2n\alpha^2+(\xi\alpha)\omega}\right] \quad (5.10)$$

Proof. Replacing Y and V by X in (5.6) and (5.9), and then taking the difference, we get

$$B(X)-C(X)=\left(\frac{-B(\xi)}{2n\alpha^2+(\xi\alpha)\omega}\right)\left[2n\alpha^2\eta(X)-(X\alpha)\omega-\phi(X)\alpha\right]$$
$$-\left(\frac{-C(\xi)}{2n\alpha^2+(\xi\alpha)\omega}\right)\left[2n\alpha^2\eta(X)-(X\xi)\omega-\phi(X)\alpha\right] \quad (5.11)$$

after simplification (5.11), we get (5.10). This completes the proof of the theorem 5.4.

If α is constant, then the relation (5.10) of Theorem 5.4 is becomes that

$$B(X)-C(X)=\left[C(\xi)-B(\xi)\right]\eta(X) \quad (5.12)$$

And putting X=ϕX in (5.12), we get

$$B(\phi X)-C(\phi X)=0$$

Remark: As a geometrical meaning of the 1-forms B and C, it is proved that “If a weakly Lorentzian Para (LP) α-Sasakian manifold $\left(M^{2n+1}, g\right)$ (n>1) is ϕ Ricci Symmetric with α is constant, then the relation

$$B(\phi X)-C(\phi X)=0$$

holds.”

Example 6.1 We consider the 3-dimensional manifold $M^3=\{(x,y,z)\in R^3, z\neq 0\}$ where (x, y, z) are the standard coordinates in R^3. Let $\{e_1, e_2, e_3\}$ be a linearly independent global frame on M given by

$$e_1=e^z\frac{\partial}{\partial x},\ e_2=e^{z-ax}\frac{\partial}{\partial y},\ e_3=\alpha\frac{\partial}{\partial z},$$

are linearly independent at each point of M where α is non zero constant. Let g be the Para Sasakian metric defined by

$$g(e_1,e_3)=g(e_1,e_2)=g(e_2,e_3)=0,$$
$$g(e_1,e_1)=g(e_2,e_2)=1, g(e_3,e_3)=-1.$$

Let $e_3=\xi$ and the Lorentzian Para α-Sasakian metric g is thus given by

$$g=e^{-2z}(dx)^2+e^{2(ax-z)}(dy)^2+\frac{1}{\alpha^2}(dz)^2$$

$$g = \begin{pmatrix} e^{-2z} & 0 & 0 \\ 0 & e^{2(ax-z)} & 0 \\ 0 & 0 & \frac{1}{\alpha^2} \end{pmatrix}$$

Let η be the 1-form defined by

$\eta(X)= g(X,\xi)$

For any vector field X on M^3. Let ϕ be the tensor field of type (1, 1) defined by

$$\phi(e_1) = -e_1, \phi(e_2) = -e_2, \phi(e_3) = 0$$

Using the linearity properties of g and ϕ, one can deduce,

$$\phi^2 X = X + \eta(X)\xi, \eta(\xi) = -1, g(\xi,\xi) = -1$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y)$$

Also

$$\eta(e_1) = 0, \eta(e_2) = 0, \eta(e_3) = -1$$

Hence for $e_3 = \xi$, (ϕ, ξ, η, g) defines a Lorentzian almost contact metric structure on M. Let ∇ be the Levi-civita connection with respect to Lorentzian metric g. then we have

$$[e_1, e_2] = -ae^z e_2, [e_1, e_3] = -\alpha e_1, [e_2, e_3] = -\alpha e_2$$

Using Koszule's formula for Levi-civita connection ∇ with respect to g, that is

$$2g(\nabla_X Y, Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y])$$

One can easily calculate

$$\nabla_{e_1} e_3 = -\alpha e_1, \nabla_{e_3} e_3 = 0, \nabla_{e_2} e_3 = -\alpha e_2$$

$$\nabla_{e_2} e_2 = -ae^z e_1 - \alpha e_3, \nabla_{e_1} e_2 = 0, \nabla_{e_2} e_1 = ae^z e_2$$

$$\nabla_{e_1} e_1 = -\alpha e_3, \nabla_{e_3} e_2 = 0, \nabla_{e_3} e_1 = 0$$

With these information the structure (ϕ, ξ, η, g) satisfies (2.3) and (2.4). Hence

$M^3(\phi,\xi,\eta,g)$ defines a Lorentzian-Para α-Sasakian manifold.

Hence the structure (ϕ,ξ,η, g) is a Lorentzian-Para α-Sasakian manifold. Now using the above results, we obtain

$$R(e_1,e_2)e_3 = 0, R(e_2,e_3)e_3 = -\alpha^2 e_2, R(e_1,e_3)e_3 = -\alpha^2 e_1$$

$$R(e_1,e_2)e_2 = (\alpha^2 - a^2 e^{2z})e_1, R(e_2,e_3)e_2 = -a\alpha e^z e_1 - \alpha^2 e_3, R(e_1,e_2)e_1 = -(\alpha^2 - a^2 e^{2z})e_2$$

$$R(e_3,e_1)e_1 = \alpha^2 e_3, R(e_2,e_1)e_1 = (\alpha^2 - a^2 e^{2z})e_2, R(e_3,e_2)e_2 = a\alpha e^z e_1 + \alpha^2 e_3$$

$$R(e_3,e_2)e_3 = \alpha^2 e_2, R(e_1,e_3)e_1 = -\alpha^2 e_3, R(e_3,e_1)e_3 = \alpha^2 e_1$$

and the components which can be obtained from these by the symmetry properties.

Using the components of the curvature tensor one can easily obtain the non vanishing components of the Ricci tensor S.

$$S(e_1,e_1) = -a^2 e^{2z}, S(e_2,e_2) = -a^2 e^{2z}, S(e_3,e_3) = -2\alpha^2$$

Since $\{e_1, e_2, e_3\}$ is an orthonrmal basis of (M^3, g) any vector X and Y can be written as

$$X = a_1e_1 + a_2e_2 + a_3e_3, Y = b_1e_1 + b_2e_2 + b_3e_3$$

Where $a_i, b_i (i = 1, 2, 3)$ are positive real numbers. Now

$$\begin{aligned} S(X, Y) &= a_1b_1S(e_1, e_1) + a_2b_2S(e_2, e_2) + a_3b_3S(e_3, e_3) \\ &+(a_1b_2 + a_2b_1)S(e_1, e_2) + (a_1b_3 + a_3b_1)S(e_1, e_3) \\ &+(a_2b_3 + a_3b_2)S(e_2, e_3) \\ &= -a_1b_1a^2e^{2z} - a_2b_2a^2e^{2z} - 2a_3b_3\alpha^2 = \sigma_1 \quad \text{Say.} \end{aligned} \tag{6.1}$$

We choose $a_i, b_i (i = 1, 2, 3)$ in such a way that $S(X, Y) = \sigma_1 \neq 0$.

We know that,

$$QX = \sum_{i=1}^{3} R(X, e_i)e_i$$

From which

$$Qe_1 = -a^2e^{2z}e_1, Qe_2 = -a^2e^{2z}e_2, Qe_3 = a\alpha e^z e_1 + 2\alpha^2 e_3$$

From the known formula,

$$(\nabla_X Q)(Y) = \nabla_X (QY) - Q(\nabla_X Y)$$

and lengthy calculations one obtains,

$$g\left(\phi^2 \left(\nabla e_1 Q\right)(X), Y\right) = \left(a_1 a\alpha^2 e^z - 2a_3\alpha^3 + 2a_1\alpha^3\right)b_1 = \sigma_2 \quad \text{Say} \tag{6.2}$$

$$\begin{aligned} g\left(\phi^2 \left(\nabla e_2 Q\right)(X), Y\right) &= \left(a_2a^3e^{3z} - a^3e^{3z} + a\alpha^2e^z\right)b_1 \\ &+\left(a_3a^2\alpha e^{2z} - a_1a^3e^{3z} - 2a_3\alpha^3 + a^3e^{3z} - \alpha a^2e^{2z}\right)b_2 = \sigma_3 \quad \text{Say.} \end{aligned} \tag{6.3}$$

$$\begin{aligned} g\left(\phi^2 \left(\nabla e_3 Q\right)(X), Y\right) &= \left(-2a_1a^2\alpha e^{2z} + a_3a\alpha^2e^z\right)b_1 \\ &-2a_2a^2\alpha e^{2z}b_2 = \sigma_4 \quad \text{Say.} \end{aligned} \tag{6.4}$$

Using (1.1), we get

$$g\left(\phi^2 \left(\nabla e_1 Q\right)(X), Y\right) = A(e_1)S(X, Y) + B(X)S(e_1, Y) + C(Y)S(X, e_1) \tag{6.5}$$

$$g\left(\phi^2 \left(\nabla e_2 Q\right)(X), Y\right) = A(e_2)S(X, Y) + B(X)S(e_2, Y) + C(Y)S(X, e_2) \tag{6.6}$$

$$g\left(\phi^2 \left(\nabla e_3 Q\right)(X), Y\right) = A(e_3)S(X, Y) + B(X)S(e_3, Y) + C(Y)S(X, e_3) \tag{6.7}$$

Consider

$$A(e_1)=\frac{\sigma_2}{\sigma_1},\ A(e_2)=\frac{\sigma_3}{\sigma_1},\ A(e_3)=\frac{\sigma_4}{\sigma_1}, \quad (6.8)$$

from (6.5), (6.6), and (6.7) it is easy to see that,

$$B(X)S(e_1,Y)+C(Y)S(X,e_1)=0 \quad (6.9)$$

$$B(X)S(e_2,Y)+C(Y)S(X,e_2)=0 \quad (6.10)$$

$$B(X)S(e_3,Y)+C(Y)S(X,e_3)=0 \quad (6.11)$$

From these homogeneous equations in B and C, one obtains non trivial solutions

$$B(e_1)=-C(e_1)$$

Hence we can take arbitrarily as,

$$B(e_1)=2a_1b_2a^2\alpha e^{2z},\ C(e_1)=-2a_1b_2a^2\alpha e^{2z} \quad (6.12)$$

Also from (6.9), (6.10) and (6.11) we find

$$B(e_2)=0, C(e_2)=0, B(e_3)=0, C(e_3)=0 \quad (6.13)$$

From 1-forms given by (6.8), (6.12), (6.13), the manifold under consideration is weakly ϕ-Ricci symmetric Lorentzian-Para α-Sasakian manifold.

This leads to the following:

Theorem 6.1 There exists a Lorentzian-Para α-Sasakian manifold (M^3, g) which is weakly ϕ-Ricci symmetric but neither locally ϕ-Ricci symmetric nor ϕ-Ricci recurrent.

Proof: Follows from (2.12) and (2.13).

References

1. Abolfazi Taleshian, Nadar Asghari, On LP-Sasakian manifolds, of *Mathematical Analysis and applications, 3(1),* 45-51 (2011).
2. K. Matsumoto, On Lorentzian Para contact manifolds, *Bull. Of Yamagata Uni. Nat. Sci. 12*, 151-156 (1989).
3. K. Matsumoto and I. Mihai, On a certain transformation in Lorentzian Para Sasakian manifold, *Tensor N.S. 47*, 187-197 (1988).
4. Lovejoy S.K. Das and M. Ahmed, CR-submanifolds LP-Sasakian manifold with a quarter symmetric and nonmetric connections, *Math. Sci. Res. J. 13*, 161-169 (2009).
5. Lovejoy S. K. Das, M. Ahmed and J. P. Ojha, CR-submanifolds LP-Sasakian manifold with a quarter symmetric and nonmetric connection,(Communicated)
6. Mobin Ahmed and Janarthan Ojha, CR-submanifolds of a Lorentzian Para Sasakian manifolds with the canonical semi symmetric metric connection, *Int. J. of Contemporary Math. Sci. 5(33),* 636-643 (2009).
7. Mahai. I. and Rosca, On Lorentzian manifolds, Classical Analysis world Scientific Pable (1992).
8. Shaikh A.A. and Biswas. S., On LP-Sasakian manifolds, *Bull. Malesian Sci. Soc. 27*, 17-26 (2004).
9. U.C. De and Anupkumar Sengupta CR-submanifolds of Lorentzian Para Sasakian manifolds, *Bull. Malesian mathematical Sci. Soc. 23*, 99-106 (2000).
10. S.S. Pujar Lorentzian-Para α-Sasakian manifold "Lorentzian-Para α-Sasakian manifold, *International Journal of Contemporary Mathematical Sciences"* (accepted 2011).
11. Shaikh A. A. and Hui S. K. On weak symmetries of trans- Sasakian manifolds, *Proceedings of Estonian Aca. Sci., 58,* 4, 213 (2009).
12. Kim J. S. *et. al.* "On generalized Ricci recurrent trans-Sasakian manifolds", *Korean Math, Soc., 39(6),* 953 (2002).