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Extrapolation method in multiparameter eigenvalue problems

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Abstract

In solving multiparameter eigenvalue problem in the form of ordinary differential equations satisfying certain boundary conditions by shooting method, good starting values of the eigenvalues to be determined are always in need. Unlike trial and error method, in this paper we present a numerical procedure based on extrapolation method to find the starting values.

Keywords and phrases : eigenvalue, eigenfunction, multi-parameter, extrapolation, shooting method.

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Introduction

Multiparameter (eigenvalue) problems had started to draw attention of the mathematicians since the unifying work of Atkinson[1]. Accounts of the topic may also be found in Atkinson[1], Baruah[2], Mena[18], Sleeman[20]. One-parameter (eigenvalue) problems are much developed, both theoretically and numerically. Compared to theoretical development, numerical development even in case of two-parameter (eigenvalue) problems are very limited. Numerical works on two-parameter problems may be found in Baruah[2,3,4,5], Baruah and Konwar[6,7], Baruah and Changmai[8,9,10,11,12,13], Brown and Sleeman[14], Fox *et al.*[15] and Gregory and Wilkerson[16]. Keeping this in mind, we therefore consider a two-parameter problem for investigation in the present work.

In the simplest form the problem is expressed mathematically[17] in the form of the second order ordinary differential equation

$$y''(x) + \{\lambda + \mu f(x) + g(x)\} y(x) = 0, \qquad (1)$$

satisfying the boundary conditions

$$y(a) = y(b) = y(c) = 0, \qquad (2)$$

where $a < b < c$, $f(x)$ and $g(x)$ are given real valued continuous functions of the independent variable $x \in [a, c]$, $f(x)$ is monotonic on $[a, c]$, the three points a, b, c are specified. Corresponding to a pair (λ, μ) of the parameters λ and μ, called an eigenvalue of the problem (1)–(2), a non-trivial solution $y(x)$ of (1)–(2), called an eigenfunction of the problem (1)-(2). For an account of the existence of eigenvalue pairs, we may cite works of Baruah[2] and Mena[18]. Fox *et. al*. have used shooting method (Roberts and Shipman[19]) to estimate eigenvalue pairs (λ, μ) of (1)–(2), the solutions corresponding to which cross the axis of x, *i.e.* vanishes the requisite number of times in the intervals [a, b] and [b, c]. The points at which the eigenfunction vanishes in [a, b] and [b, c] are called zeros of the eigenfunction. The eigenfunction is said to be oscillatory in [a, b] or in [b, c], if there exists at least one zero in the interval. By virtue of the boundary conditions imposed we must ensure of eigenfunctions which are at best oscillatory in the present case, the zeros being a and b in [a, b] as well as b and c in [b, c].

However, to start with the shooting method one needs starting values of the entities λ and μ whose actual values are to be determined. The convergence of the shooting method and, if it exists, its rapidity both depend on the starting values. In finding actual values of the eigenvalue pairs (λ, μ) by shooting method, Fox *et. al*.[15] have used their starting values obtained by a trial and method. In this paper we present a numerical procedure based on Extrapolation method in finding starting values for the shooting method. For actual values of certain eigenvalue pairs (λ, μ), the number of zeros (m, n) of the corresponding eigenfunctions in each of the intervals (a, b) and (b, c) can be computed. A numerical example is also given.

Formulation of the Procedure :

The numerical procedure is based on finding first few values of the function $y(x)$ and its second derivative $y''(x)$ of the problem (1)-(2), whence a numerical formula for integrating ahead by extrapolation (Hildebrand[17]) could be quite useful in finding starting values of the eigenvalue pairs (λ, μ) of (1)-(2). For the second order differential equation (1) which do not contain first derivative and satisfy the initial conditions

$$y(a) = 0, \quad y'(a) = 1, \qquad (3)$$

the second condition merely providing a particular normalization of the eigenfunction to be computed, the formula for integrating ahead by extrapolation in finding the step-by-step solutions of (1)-(3) can be expressed as

$$y_{r+1}=2y_r-y_{r-1}+h^2[y''_r+1/12(\Delta_2 y''_r+\Delta_3 y''_r+\Delta_4 y''_r)]. \qquad (4)$$

Here h denotes the interval between the equidistant values of x such that the equations

$$ih = b - a, \quad jh = c - b \qquad (5)$$

can be satisfied for integers i and j and for any integer r, the value of y_r is given by

$$y_r = y(a + rh). \qquad (6)$$

Along with the first few values of the function $y(x)$, the formula (4) involve the first four differences of the function $y''(x)$.

In order to achieve our objectives, now we derive a truncated Taylor formula for $y(x)$ about the point $x = a$. For this purpose we write the differential equation (1) in the form :

$$y''(x) = -\{\lambda + \mu f(x) + g(x)\}\, y(x) = 0. \qquad (7)$$

Using the initial conditions (3) and successive differentiation of (7), we obtain the values of $y(x)$ and its successive derivatives at $x = a$ as

$y(a)=0$, $y'(a)=1$, $y''(a)=0$, $y'''(a) = -\{\lambda + \mu f(a) + g(a)\}$

$y^{iv}(a) = -2\{\mu f'(a) + g'(a)\}$, and

$$y^{v}(a) = \{\lambda + \mu f(a)+g(a)\}^2 - 3\{\mu f''(a)+g''(a)\}. \quad (8)$$

In the above set of equations the values of $f(x)$ and $g(x)$ and the values of their derivatives at $x = a$ are known and hence the values of (8) are known. The Taylor formula of $y(x)$ about $x = a$ is then given by

$$y(x) = x - a - 1/3!\{\lambda + \mu f(a) + g(a)\}(x-a)^3 - 2/4!\{\mu f'(a) + g'(a)\}(x-a)^4 + 1/5!\{\{\lambda + \mu f(a) + g(a)\}^2 - 3\{\mu f''(a) + g''(a)\}\}(x-a)^5 + \ldots \quad (9)$$

In finding the values of $y(x)$ using the above formula (9), it is desirable to keep $|x - a|$ numerically small in order to have rapid convergence of the series so as to obtain higher accuracy in the values of $y(x)$. Hence in general we should work on both sides of $x = a$, that is, we shall compute values of $y(x)$ on both right and left sides of the point $x = a$.

Now, on using equations (5) and (6) we have

$$y_{-2} = y(a-2h),\quad y_{-1} = y(a-h),\quad y_0 = y(a),$$
$$y_1 = y(a+h),\quad y_2 = y(a+2h),\quad y_3 = y(a+3h),$$
$$y_i = y(a+ih) = y(b),\ \ldots,\ y_j = y(a+jh) = y(c). \quad (10)$$

By virtue of the boundary conditions (2), further we have

$$y_0 = y_i = y_j = 0. \quad (11)$$

The first five values of $y(x)$, namely, y_{-2}, y_{-1}, y_0, y_1 and y_2 are given by the Taylor formula (9) corresponding to the values of x as $a-2h$, $a-h$, a, $a+h$ and $a+2h$ respectively. The differential equation (7) then give the corresponding first five values of $y''(x)$ in terms of λ and μ as :

$$y''_{-2} = -\{\lambda+\mu f(a-2h) + g(a-2h)\}\, y_{-2},$$
$$y''_{-1} = -\{\lambda+\mu f(a-h) + g(a-h)\}\, y_{-1},$$
$$y''_0 = -\{\lambda+\mu f(a)+g(a)\}\, y_0,$$
$$y''_1 = -\{\lambda+\mu f(a+h) + g(a+h)\}\, y_1, \text{ and}$$
$$y''_2 = -\{\lambda + \mu f(a+2h) + g(a+2h)\}\, y_2, \quad (12)$$

where by virtue of (11), $y''_0 = 0$. For the five values of y''_{-2}, y''_{-1}, y''_0, y''_1 and y''_2, the corresponding differences in terms of λ and μ are

$$\Delta_1 y''_{-1} = y''_{-1} - y''_{-2},\ \Delta_1 y''_0 = y''_0 - y''_{-1},\ \Delta_1 y''_1 = y''_1 - y''_0, \text{ and } \Delta_1 y''_2 = y''_2 - y''_1.$$
$$\Delta_2 y''_0 = \Delta_1 y''_0 - \Delta_1 y''_{-1},\ \Delta_2 y''_1 = \Delta_1 y''_1 - \Delta_1 y''_0, \text{ and } \Delta_2 y''_2 = \Delta_1 y''_2 - \Delta_1 y''_1.$$
$$\Delta_3 y''_1 = \Delta_2 y''_1 - \Delta_2 y''_0, \text{ and } \Delta_3 y''_2 = \Delta_2 y''_2 - \Delta_2 y''_1.$$
$$\Delta_4 y''_2 = \Delta_3 y''_2 - \Delta_3 y''_1. \quad (13)$$

The formula (4) then gives the first extrapolated value $y_3 = y(a+3h)$ as

$$y_3 = 2y_2 - y_1 + h^2[y''_2 + 1/12(\Delta_2 y''_2 + \Delta_3 y''_2 + \Delta_4 y''_2)], \quad (14)$$

where the right hand side of the equation (14) in terms of λ and μ is known. For y_3, the given differential equation (7) gives the value of y''_3 as

$$y''_3 = -\{\lambda + \mu f(a+3h) + g(a+3h)\}\, y_3, \quad (15)$$

whence we have

$$\Delta_1 y''_3 = y''_3 - y''_2,\quad \Delta_2 y''_3 = \Delta_1 y''_3 - \Delta_1 y''_2,$$
$$\Delta_3 y''_3 = \Delta_2 y''_3 - \Delta_2 y''_2, \text{ and } \Delta_4 y''_3 = \Delta_3 y''_3 - \Delta_3 y''_2. \quad (16)$$

From formula (4), the second extrapolated value $y_4 = y(a+4h)$ is then given by

$y_4 = 2y_3 - y_2 + h^2 [y''_3 + 1/12(\Delta_2 y''_3 + \Delta_3 y''_3 + \Delta_4 y''_3)].$ (17)

The above process of finding the second derivatives of the extrapolated values of y's from equation (7) and subsequent evaluation of the next extrapolated values of y's using the formula (4) at each step are continued until y_i and y_j are obtained. On equating both y_i and y_j to zero as given by (11), the corresponding solutions will then give the starting values of the eigenvalue pairs (λ, μ) for shooting method of the problem (1)-(2).

A Numerical Example :

We consider the numerical example formed by the differential equation

$(1 + x + x^2)\, y''(x) + \{\lambda+\mu\, x+x^2\}\, y(x)=0$ (18)

subject to the boundary conditions

$y(-1) = y(0) = y(1) = 0. \quad [\, y'(-1) = 1\,]$ (19)

Here $a = -1$, $b = 0$, $c = 1$.

Following the approach of obtaining the equations (8), we get

$y(-1)=0,\ y'(-1)=1,\ y''(-1)=0,\ y'''(-1)=\mu-\lambda-1$
$y^{iv}(-1) = 2 - 2\lambda$, and $y^{v}(-1) = \lambda^2 + \mu^2 - 2\lambda\mu + 2\lambda - 8\mu + 7.$ (20)

The Taylor formula about the point $x = -1$ is then given by

$y(x) = x + 1 + 1/3!\, (\mu - \lambda - 1)(x + 1)^3 + 1/4!\, (2 - 2\lambda)(x + 1)^4 + 1/5!\, (\lambda^2+\mu^2 - 2\lambda\mu+2\lambda - 8\mu+7)(x + 1)^5+\ldots$ (21)

For $a = -1$ and choosing $h = 0.2$ in (10), here we have

$y_{-2} = y(-1.4)$, $y_{-1} = y(-1.2)$, $y_0 = y(-1.0)$,
$y_1 = y(-0.8)$, $y_2 = y(-0.6)$, $y_3 = y(-0.4)$,
$y_4 = y(-0.2)$, $y_5 = y(0.0)$, $y_6 = y(0.2)$,
$y_7 = y(0.4)$, $y_8 = y(0.6)$, $y_9 = y(0.8)$
and $y_{10} = y(1.0)$, (22)

where by virtue of the boundary conditions (19) we obtain

$y_0 = y_5 = y_{10} = 0. \quad (i = 5,\ j = 10)$ (23)

Using the Taylor formula (21) we now obtain:

$y_{-2} = -0.3879 + 0.0084\lambda - 0.0099\mu - 0.0001\lambda^2 + 0.0002\lambda\mu - 0.0001\mu^2$
$y_{-1} = -0.1986 + 0.0012\lambda - 0.0013\mu$
$y_0 = 0$
$y_1 = 0.1988 - 0.0014\lambda + 0.0013\mu$
$y_2 = 0.3921 - 0.0126\lambda + 0.0099\mu + 0.0001\lambda^2 - 0.0002\lambda\mu + 0.0001\mu^2$ (24)

Applying the procedure described in the previous section, we now obtain the extrapolated values of y's as below :

$y_3 = 0.5784 - 0.0441\lambda + 0.0301\mu + 0.0010\lambda^2 - 0.0014\lambda\mu + 0.0006\mu^2$
$y_4 = 0.7599 - 0.1052\lambda + 0.0617\mu + 0.0043\lambda^2 - 0.0051\lambda\mu + 0.0017\mu^2 - 0.0001\lambda^3 + 0.0001\lambda^2\mu - 0.0001\lambda\mu^2$
$y_5 = 0.9396 - 0.2018\lambda + 0.1003\mu + 0.0126\lambda^2 - 0.0126\lambda\mu + 0.0033\mu^2 - 0.0004\lambda^3 + 0.0005\lambda^2\mu - 0.0004\lambda\mu^2$
$y_6 = 1.1189 - 0.3357\lambda + 0.1390\mu + 0.0289\lambda^2 - 0.0239\lambda\mu + 0.0049\mu^2 - 0.0012\lambda^3 + 0.0014\lambda^2\mu - 0.0008\lambda\mu^2$

$$y_7 = 1.2966 - 0.5051\,\lambda + 0.1704\,\mu + 0.0559\,\lambda^2 - 0.0374\,\lambda\mu + 0.0056\,\mu^2 - 0.0029\,\lambda^3 + 0.0029\,\lambda^2\mu - 0.0012\,\lambda\mu^2$$

$$y_8 = 1.4989 - 0.7056\,\lambda + 0.1879\,\mu + 0.0955\,\lambda^2 - 0.0499\,\lambda\mu + 0.0046\,\mu^2 - 0.0060\,\lambda^3 + 0.0047\,\lambda^2\mu + 0.0013\,\lambda\mu^2 - 0.0001\,\mu^3 + 0.0001\,\lambda^4$$

$$y_9 = 1.6304 - 0.9308\,\lambda + 0.1861\,\mu + 0.1487\,\lambda^2 - 0.0572\,\lambda\mu - 0.0059\,\mu^2 - 0.0110\,\lambda^3 + 0.0063\,\lambda^2\mu - 0.0009\,\lambda\mu^2 - 0.0003\,\mu^3 + 0.0003\,\lambda^4$$

$$y_{10} = 1.7748 - 1.1729\,\lambda + 0.1611\,\mu + 0.2155\,\lambda^2 - 0.0548\,\lambda\mu - 0.0187\,\mu^2 - 0.0183\,\lambda^3 + 0.0068\,\lambda^2\mu + 0.0004\,\lambda\mu^2 - 0.0004\,\mu^3 + 0.0007\,\lambda^4 - 0.0001\,\lambda^2\mu^2 \quad (25)$$

Now, by virtue of the boundary conditions (23), equating both y_5 and y_{10} *i.e.*, y(0) and y(1) to zero and solving them as simultaneous equations we find a value pair of λ and μ as

$$(\lambda, \mu) \approx (9.0, 7.7). \quad (26)$$

There are other solutions of the simultaneous equations which we did not compute.

Finally, on using the above value pair (26) as the starting value for the shooting method of the problem (18)-(19), we find the actual value of the eigenvalue pair (λ, μ) as

$$(\lambda, \mu) = (11.25055, 3.86703), \quad (27)$$

the corresponding eigenfunction of which possess no internal zeros in both (–1, 0) and (0, 1).

Conclusion

For shooting method of the problem (1)-(2), starting values of the eigenvalue pairs (λ, μ) of the problem (1)-(2) are obtained as solutions of the simultaneous equations given by (11). Moreover, a systematic mathematical procedure described here is more logical than a trial and error method used normally for the purpose.

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