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On pseudo fuzzy complements of fuzzy sets and the existence of equilibrium and dual points

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Abstract

Fuzzy sets are sets with boundaries that are not precise. Fuzzy complements are fuzzy sets on [0,1] satisfying two axioms,viz. boundary conditions and monotonicity. But the existence of an equilibrium point of a fuzzy complement depends on its continuity. Here we introduce a new concept: pseudo fuzzy complement and prove that all such complements have a unique equilibriumpoint and dual point.

Key words : Fuzzy set, Fuzzy complement, Equilibrium point, Fuzzy pseudo complement, Dual point.

1. Introduction

An important event in the evolution of the modern concept of uncertainty was the publication of the seminar paper entitled "Fuzzy sets" by L.A. Zadeh[4] in 1965. The significance of Zadeh's paper was that it challenged not only probability theory as the only tool for studying uncertainty, but also the very foundation on which probability theory is based viz. 2-valued logic[2].

In the usual set theory introduced and developed by G. Cantor, an element may or may not belong to a set. No other intermediate possibility is allowed. Hence the usual set theory is not capable of handling several ambiguous real-life situations. Zadeh's notion "fuzzy sets" allows all levels of membership between "non-membership" and "membership", including both. Hence it rectifies the above mentioned defect of usual sets. More formally, a fuzzy set A on a universal set X is defined as a function $A : X \rightarrow [0,1]$. This definition generalises the notion of characteristic functions of ordinary sets.

Fuzzy complements are fuzzy sets on [0,1] satisfying two axioms, *viz*. boundary conditions and monotonicity. But the existence of an equilibrium point of a fuzzy complement depends on its continuity. Here we introduce a new concept: pseudo fuzzy complement and prove that all such complements have a unique equilibrium point and dual point.

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2.*Equilibrium Point*[2] *(or Fixed Point)*

An ***equilibrium point (****or a* ***fixed point)*** of a mapping T: X→X of a set X in to itself is an element x ε X which is mapped onto itself. That is, an element x ε X is said to be an equilibrium point of T if T(x) = x. We shall write T(x) as T_x.

3. *Example.* (1) Let X be a nonempty set and I : X → X be the identity function. Then every element of X is an equilibrium point of I.
(2) Let R denote the real number system. Then the mapping T:R→R defined by Tx = x^2 has two fixed points : 0 and 1.

4. *Contraction*[4] Let (X,d) be a metric space. A mapping T:X→X is called a ***contraction*** on X if there is a positive real number $\alpha < 1$ such that for all x, y ε X

$$d(Tx, Ty) \leq \alpha\, d(x, y)$$

Following the usual practice we shall denote a metric space (X, d) by X.

5. *Banach Fixed Point Theorem or Contraction Theorem*[1]

Suppose that X is a complete metric space and T:X→X is a contraction on X. Then T has precisely one fixed point.

6. *Fuzzy Set*[2,4]

The characteristic function of a *crisp set* (*classical set* or *ordinary set*) assigns a value of either 1 or 0 to each individual element in the universal set, thereby discriminating between members and non-members of the crisp sets under consideration. This function can be generalized in such a way that the values assigned to the elements of the universal set fall within a specified range and indicate the membership grade of these elements in the set in question. Larger values denote the higher degrees of the set membership. Such a function is called a ***membership function***, and the set defined by it a ***fuzzy set***.Themost commonly used range of values of membership functions is the unit interval **[0,1].**

7. *Example.* Consider a fuzzy set corresponding to the concept "Temperature" of the cities Jaipur, New Delhi, Mumbai, Chennai and Kochi. Let the following be the temperatures of these cities at a given point of time.

	Min. Temp.	Max. Temp.
Jaipur	28	38
New Delhi	20	32
Mumbai	24	30
Chennai	26	34
Kochi	22	28

Let μ be the fuzzy set corresponding to the concept **"High Temperature".** Then we can write μ as

μ ={(Jaipur,1), (New Delhi, 0.7), (Mumbai, 0.6), (Chennai, 0 .8), (Kochi, 0.5)}

Let β be the fuzzy set corresponding to the concept **"Low Temperature".**
Let β={ (Jaipur, 0.6), (New Delhi,1), (Mumbai, 0.8), (Chennai, 0.7), (Kochi, 0.9)}

8. *Remark :*

From the above defined fuzzy sets, the membership grades (grade values) corresponding to the cities gives an idea about the corresponding fuzzy concepts.

From the fuzzy set μ, we can say that Jaipur has the highest temperature (because its grade value 1 is the maximum) and that of Kochi is the lowest in this category (because its grade 0.5 is the lowest).

Similarly from the fuzzy set β, we can say that New Delhi has the lowest temperature (because its grade 1 is the maximum) and that of Jaipur has the lowest in this category (because its grade 0.6 is the lowest).

9. *Fuzzy Complements*[2]

A function $c:[0,1]\to[0,1]$ is called a ***fuzzy complement*** if it satisfy the following axioms;

Axiom *c*1 : $c(0) = 1$ and $(1) = 0$ (Boundary conditions)

Axiom *c*2 : For all a, b ε [0, 1],if a ≤ b, then $(a) \geq c(b)$ (Monotonicity)

The axioms ***c*1** and ***c*2** are jointly called the *axiomatic skeleton for fuzzy complements.*

10. *Example*. Let 't' be a fixed point in [0, 1). Define a function : $[0,1] \to [0,1]$ by c(a) = 1, for a ≤ t and c(a) = $\dfrac{a-1}{t-1}$, for a t. the c satisfies the axioms ***c*1** and ***c*2;** hence c is fuzzy complement.

11. *Remark*. Two of the most desirable additional requirements for fuzzy complements which are usually listed in the literature are the following:

Axiom *c*3 : is a continuous function.

Axiom *c*4 : is involutive, which means that $c(c(a)) = a$, for each a ε [0,1]

12. *Example.* The function c: $[0, 1] \to [0, 1]$ defined by $c(a) = (1+\cos \pi a)/2$ is a continuous fuzzy complement but not involutive.

13. *Example.* For λ ε (-1, ∞), t he function $c_\lambda: [0, 1] \to [0, 1]$ defined by $c_\lambda(a)=\dfrac{1-a}{1+\lambda a}$ is an involutive fuzzy complement. The above example presents an infinite class of fuzzy complements, called the Sugeno class of fuzzy complements.

14. *Theorem*[2]

Every fuzzy complement has at most one equilibrium

15. *Theorem*[2]

If *c* is a continuous fuzzy complement, then *c* has a unique equilibrium.

16. *Dual point*[3]

Let c: $[0, 1] \to [0,1]$ be a fuzzy complement and a ε [0,1]. Then an element d_aε [0,1] is called a *dual point* of 'a' with respect to 'c' if $c(d_a) - d_a = a - c(a)$.

17. *Example.* Each one of the Sugeno Class of fuzzy complements $c_\lambda(a)$ provides a unique dual point to each element a ε [0,1].

18. *Theorem*[2]

If a complement c has an equilibrium, e_c, then $d_{e_c} = e_c$

19. *Theorem*[2]

For each a ε [0,1], $d_a = c(a)$ iff $c(c(a)) = a$, that is, when the complement is involutive.

20. *Remark.* It follows from theorem .15 that a dual point exists for each a ε [0,1] when 'c' is a continuous complement.

21. *Pseudo Fuzzy Complement*.A function c:[0,1] $\rightarrow$ [0,1] is called a ***pseudofuzzy complement*** if it satisfy the following axioms:

Axiom *c*1' : $c(0)$ = k and $c(1)$=0, where k < 1
Axiom *c*2 : For all a, b ε [0,1], if a ≤ b, then $c(a) \geq c(b)$ (Monotonicity)

22. *Remark*. The difference between *fuzzy complement* and *pseudo fuzzy complement* is only in the first axiom. Here we modify the first axiom of the fuzzy complement by reducing the upper bound of c to a real number, which is less than one[1].

23. *Example.*Fix an element t ε[0,1) and define a function c : [0, 1] → [0, 1] by

$$c(a)=\begin{cases} .9, & for\ a \leq t \\ \frac{.9(a-1)}{t-1}, & for\ a\ \ t \end{cases}$$, where a ε [0,1]

Then c satisfies both the axioms $c1'$ and $c2$, so it is a pseudo fuzzy complement.

24. *Theorem.* Every pseudo fuzzy complement: [0,1] → [0,1] has a unique equilibrium point.

Proof: Let X= [0,1] and let 'd' be the usual metric on X defined by
d(x, y) = $|x-y|$, for all x, y ε [0, 1].
Then X is a complete metric space and d (x,y) ≤ 1, for all x, y ε X.
Also, from the definition of 'd'and since 'c' is a pseudo fuzzy complement, we get, for all x, y εX,
$d(c(x), c(y))=|c(x)-c(y)| \leq |k-0| \leq k$ -----(i)
Now let α be a positive real number such that α < 1. Then
$d(c(x), c(y)) \leq \alpha\, d(x, y)$ whenever $k \leq \alpha\, d(x, y)$ by (i)

i.e. whenever $\frac{k}{\alpha} \leq d(x, y) \leq 1$

i.e. whenever $k \leq \alpha$.

Thus $d(c(x), c(y)) \leq \alpha\, d(x, y)$, for every x, y ε X, whenever k ≤ α <1. But there are infinitely many α's such that k ≤ α <1. For any choice of such an α, c is a contraction on the complete metric space X. Hence, by Banach Fixed Point theorem, c has a unique equilibrium point ■

25. *Theorem*

If c : [0,1] → [0,1] is a pseudo fuzzy complement, then each a ε [0,1) has a dual point.

Proof: Follows from the remark.20 and the preceding theorem ■

26. **Conclusion**

It is well-known that a fuzzy complement 'c' may or may not have an equilibrium point. Continuity of 'c 'is required for the existence of both. In this paper, we have introduced the notion of pseudo fuzzy complement by slightly modifying axiom ***c*1** of fuzzy complement. Then we observe that Banach fixed point theorem is applicable in this case, and using it we prove that every pseudo fuzzy complement has a unique equilibrium point. Also, we prove that, in this case, each element in [0,1] has a unique dual point.

27. **References**

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